System Modeling and Simulation

Chapter 3. Solution Methods for Dynamic Models

3.1 Differential Equations

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Differential Equations

- An *ordinary differential equation (ODE)* is an equation containing ordinary, but not partial, derivatives of the dependent variable.
- E.g. $3\ddot{x}+7\dot{x}+2t^2x=5+\sin t$

  x: The dependent variable

  5+sint:input

  x(t): response

If right-hand side is zero, homogeneous; otherwise, non homogeneous.

- Initial condition: The specified value of x at t_0 is denoted x_0 and is called the *initial condition*. Often the starting time t_0 is taken to be at $t = 0$.
- E.g. $2\dot{x} + 6x = 3$

The solution: $x(t) = Ce^{-3t} + 0.5$

Using the initial condition to determine C

Classification of differential equations

- Linear differential equations: contain only linear functions of the dependent variable and its derivatives.
- E.g. $\dot{x} + 3x = 5 + t^2$ *constant-coefficient*

  $\dot{x} + 3t^2 x = 5$ *variable-coefficient* (t^2)

  $3\ddot{x} + 7\dot{x} + 2t^2 x = \sin t$

Classification of differential equations

- Nonlinear differential equations

$$2\ddot{x} + 7\dot{x} + 6x^2 = 5 + t^2$$

$$3\ddot{x} + 5\dot{x}^2 + 8x = 4$$

$$\ddot{x} + 4x\dot{x} + 3x = 10$$

Order of the equation

- The *order* of the equation is the order of the highest derivative of the dependent variable in the equation.
- The equation $3\ddot{x}+7\dot{x}+2x = 5$ is thus called a *second-order* differential equation.
- What about this one?

$$3\dot{x}_1 + 5x_1 - 7x_2 = 5$$
$$\dot{x}_2 + 4x_1 + 6x_2 = 0$$

Method to solve the equations

- Separation of variables :

$\dot{x}$=g(t)f(x) First write the equation as

$$\frac{dx}{f(x)} = g(t)dt$$ Then integrate both sides to obtain

$$\int_{x(0)}^{x(t)} \frac{1}{f(x)} dx = \int_0^t g(t)dt$$

Method to solve the equations

- Separation of variables :

Example. 1 $\dot{x}+2x=20 \quad x(0)=3$

First write the equation as:

$$\frac{dx}{dt} = 20 - 2x$$

Integrate both sides to obtain:

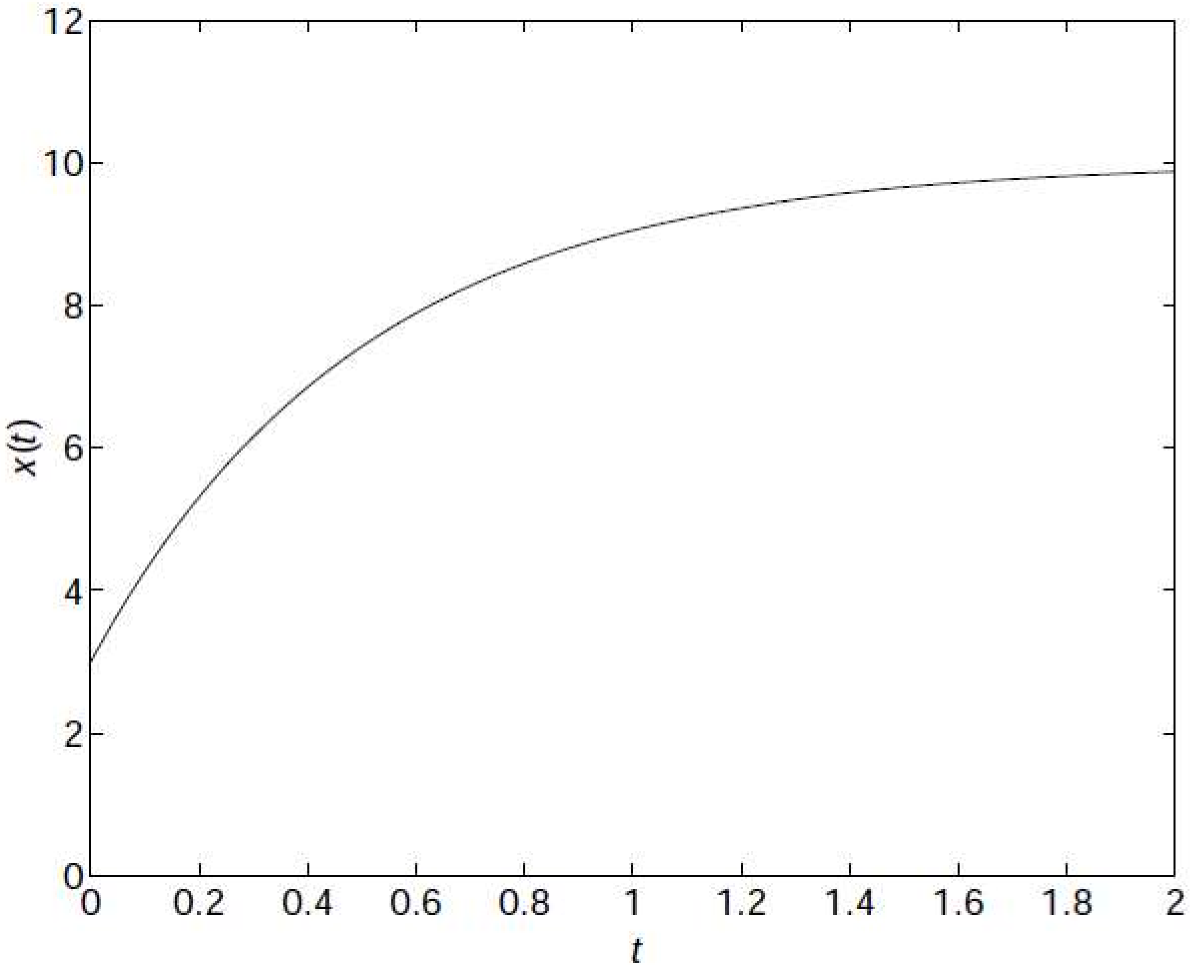
$$\int_3^{x(t)} \frac{1}{20-2x}\,dx = \int_0^t dt = t$$

The integral on the left can be evaluated as follows:

$$\ln[20-2x(t)]-\ln[20-x(3)]=-2t$$

The solution is $x(t)=10-7e^{-2t}$

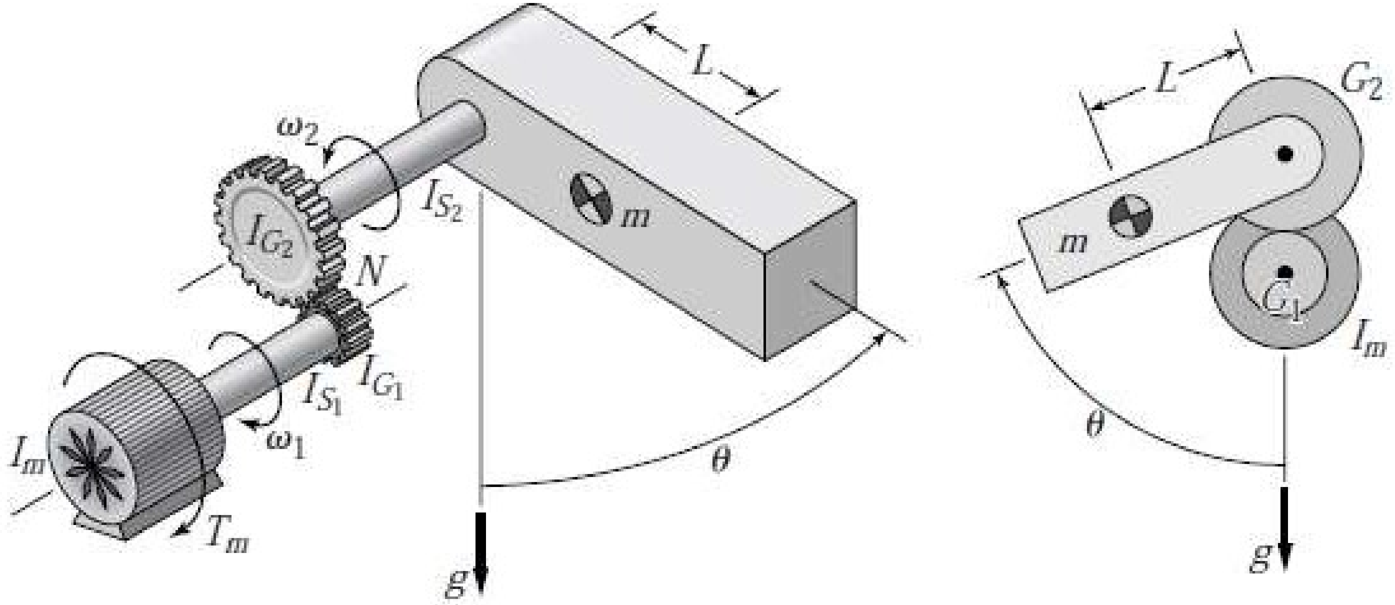
The figure of the solution as follow.



Response for Example.1

Example of Mechanical system

ot-arm



To control the motion of the arm we need to have its equation of motion. Obtain this equation in terms of the angle θ. The given values for the motor, shaft, and gear inertias are

$$I_m = 0.05\ \text{kg}\cdot\text{m}^2 \qquad I_{G_1} = 0.025\ \text{kg}\cdot\text{m}^2 \qquad I_{S_1} = 0.01\ \text{kg}\cdot\text{m}^2$$

$$I_{G_2} = 0.1\ \text{kg}\cdot\text{m}^2 \qquad I_{S_2} = 0.02\ \text{kg}\cdot\text{m}^2$$

System modeling and simulation

Chapter 3. Solution Methods for Dynamic Models

3.2 The Laplace Transform Method

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The Laplace Transform Method

The Laplace Transform Method

- Advantages:

1. Especially useful for nonhomogeneous equations

2. Converts linear differential equations into algebraic relations

- The Laplace transform *L*[*x(t)*] of a function *x(t)* is defined as follows

$$\mathcal{L}[x(t)] = \int_0^{\infty} x(t)e^{-st}\,dt \quad \text{or} \quad \mathcal{L}[x(t)] = \lim_{T\to\infty}\left[\int_0^{T} x(t)e^{-st}\,dt\right]$$

s is a complex number; *x(t)* is zero for *t* < 0.

- Notation:

$$X(s) = \mathcal{L}[x(t)] \qquad x(t) = \mathcal{L}^{-1}[X(s)]$$

Symbol $\mathcal{L}^{-1}$ denotes the *inverse* transform

- For some simple functions the Laplace transform does not exist (such as e^{t^2} or, 1/*t*).
- Transforms of common functions as follow.

$X(s)$	$x(t), t \geq 0$
1. 1	$\delta(t)$, unit impulse
2. $\frac{1}{s}$	$u_s(t)$, unit step
3. $\frac{c}{s}$	constant, c
4. $\frac{e^{-sD}}{s}$	$u_s(t-D)$, shifted unit step
5. $\frac{n!}{s^{n+1}}$	t^n
6. $\frac{1}{s+a}$	e^{-at}
7. $\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$
8. $\frac{b}{s^2+b^2}$	$\sin bt$
9. $\frac{s}{s^2+b^2}$	$\cos bt$
10. $\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$

11.	$\dfrac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
12.	$\dfrac{a}{s(s+a)}$	$1-e^{-at}$
13.	$\dfrac{1}{(s+a)(s+b)}$	$\dfrac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$
14.	$\dfrac{s+p}{(s+a)(s+b)}$	$\dfrac{1}{b-a}\left[(p-a)e^{-at}-(p-b)e^{-bt}\right]$
15.	$\dfrac{1}{(s+a)(s+b)(s+c)}$	$\dfrac{e^{-at}}{(b-a)(c-a)}+\dfrac{e^{-bt}}{(c-b)(a-b)}+\dfrac{e^{-ct}}{(a-c)(b-c)}$
16.	$\dfrac{s+p}{(s+a)(s+b)(s+c)}$	$\dfrac{(p-a)e^{-at}}{(b-a)(c-a)}+\dfrac{(p-b)e^{-bt}}{(c-b)(a-b)}+\dfrac{(p-c)e^{-ct}}{(a-c)(b-c)}$

The linearity property

- The Laplace transform is a definite integral, leads to the *linearity property* of the transform.

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s) + bG(s)$$

- The inverse transform also has the linearity property.

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)] = af(t) + bg(t)$$

- E.g. $x(\mathrm{t}) = 6 + 4e^{-3t}$

$$L(x(\mathrm{t})) = \frac{6}{s} + \frac{4}{s+3}$$

$x(t)$	$X(s) = \int_0^\infty f(t)e^{-st}\,dt$
1. $af(t) + bg(t)$	$aF(s) + bG(s)$
2. $\dfrac{dx}{dt}$	$sX(s) - x(0)$
3. $\dfrac{d^2x}{dt^2}$	$s^2X(s) - sx(0) - \dot{x}(0)$
4. $\dfrac{d^nx}{dt^n}$	$s^nX(s) - \sum_{k=1}^{n} s^{n-k} g_{k-1}$ $g_{k-1} = \left.\dfrac{d^{k-1}x}{dt^{k-1}}\right\|_{t=0}$
5. $\int_0^t x(t)\,dt$	$\dfrac{X(s)}{s} + \dfrac{g(0)}{s}$ $g(0) = \left.\int x(t)\,dt\right\|_{t=0}$
6. $x(t) = \begin{cases} 0 & t < D \\ g(t-D) & t \ge D \end{cases}$ $= u_s(t-D)g(t-D)$	$X(s) = e^{-sD}G(s)$
7. $e^{-at}x(t)$	$X(s+a)$
8. $tx(t)$	$-\dfrac{dX(s)}{ds}$
9. $x(\infty) = \lim_{s\to 0} sX(s)$	
10. $x(0+) = \lim_{s\to\infty} sX(s)$	

- Another property of the Laplace transform is called *shifting along the s-axis* or *multiplication by an exponential*. This property states that

$$\mathcal{L}[e^{-at}x(t)] = X(s+a)$$

To derive this property, note that

$$\mathcal{L}[e^{-at}x(t)] = \int_0^\infty e^{-at}x(t)e^{-st}\,dt = \int_0^\infty x(t)e^{-(s+a)t}\,dt$$
$$= X(s+a)$$

- E.g. The function te^{-at}

Here the function *x(t)* is *t*, *X(s)* = 1/*s*^{2} , and thus

$$\mathcal{L}[e^{-at}x(t)] = \mathcal{L}(te^{-at}) = \left.\frac{1}{s^2}\right|_{s\to s+a} = \frac{1}{(s+a)^2}$$

- Another property is *multiplication by t*. It states that

$$\mathcal{L}[tx(t)] = -\frac{dX(s)}{ds}$$

To derive this property, note that

$$\frac{d}{ds}X(s) = \frac{d}{ds}\left[\int_0^\infty x(t)e^{-st}\,dt\right] = -\int_0^\infty tx(t)e^{-st}\,dt = -\mathcal{L}[tx(t)]$$

- E.g. The Function $t \cos \omega t$

Here the function $x(t)$ is $\cos \omega t$, $X(s) = s/(s^2 + \omega^2)$, and thus

$$\mathcal{L}[tx(t)] = \mathcal{L}(t\cos\omega t) = -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

The Derivative Property

- To use the Laplace transform to solve differential equations, we will need to obtain the transforms of derivatives.

$$\mathcal{L}\left(\frac{dx}{dt}\right) = \int_0^\infty \frac{dx}{dt} e^{-st}\, dt = x(t)e^{-st}\Big|_0^\infty + s\int_0^\infty x(t)e^{-st}\, dt$$
$$= s\mathcal{L}[x(t)] - x(0) = sX(s) - x(0)$$

- This procedure can be extended to higher derivatives.

$$\mathcal{L}\left(\frac{d^2x}{dt^2}\right) = s^2X(s) - sx(0) - \dot{x}(0)$$

$x(t)$	$X(s) = \int_0^{\infty} f(t)e^{-st}\,dt$
1. $af(t) + bg(t)$	$aF(s) + bG(s)$
2. $\dfrac{dx}{dt}$	$sX(s) - x(0)$
3. $\dfrac{d^2x}{dt^2}$	$s^2X(s) - sx(0) - \dot{x}(0)$
4. $\dfrac{d^nx}{dt^n}$	$s^nX(s) - \sum_{k=1}^{n} s^{n-k}g_{k-1}$
	$g_{k-1} = \left.\dfrac{d^{k-1}x}{dt^{k-1}}\right\vert_{t=0}$
5. $\int_0^t x(t)\,dt$	$\dfrac{X(s)}{s} + \dfrac{g(0)}{s}$
	$g(0) = \left.\int x(t)\,dt\right\vert_{t=0}$
6. $x(t) = \begin{cases} 0 & t < D \\ g(t-D) & t \geq D \end{cases}$	
$= u_s(t-D)g(t-D)$	$X(s) = e^{-sD}G(s)$
7. $e^{-at}x(t)$	$X(s+a)$
8. $tx(t)$	$-\dfrac{dX(s)}{ds}$
9. $x(\infty) = \lim_{s\to 0} sX(s)$	
10. $x(0+) = \lim_{s\to\infty} sX(s)$	

Solving Equations With The Laplace Transform

Now show how to solve differential equations by using the Laplace transform.

$$\dot{x} + ax = f(t) \quad \Rightarrow \quad \mathcal{L}(\dot{x} + ax) = \mathcal{L}[f(t)]$$

$$\Rightarrow \quad \mathcal{L}(\dot{x}) + a\mathcal{L}(x) = \mathcal{L}[f(t)] \quad \Rightarrow \quad sX(s) - x(0) + aX(s) = F(s)$$

$$\Rightarrow \quad X(s) = \frac{x(0)}{s+a} + \frac{1}{s+a}F(s)$$

The inverse operation gives

$$x(t) = \mathcal{L}^{-1}\left[\frac{x(0)}{s+a}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+a}F(s)\right]$$

Solving Equations With The Laplace Transform

- Example: Determine the complete response of the following model: $\dot{x} + 3x = 5t \qquad x(0) = 10$

Applying the transform to the equation we obtain:

$$sX(s) - x(0) + 3X(s) = \frac{5}{s^2}$$

Solve for *X(s)*.

$$X(s) = \frac{x(0)}{s+3} + \frac{5}{s^2(s+3)} = \frac{10}{s+3} + \frac{5}{s^2(s+3)}$$

Solving Equations With The Laplace Transform

- The free response is given by the first term on the right-hand side and is $10e^{-3t}$

To find the forced response, we express the second term on the right as

$$\frac{5}{s^2(s+3)} = \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{C_3}{s+3}$$

Now use the *least common denominator (LCD)* method to obtain the coefficients *C*1, *C*2, and *C*3.

$$\frac{5}{s^2(s+3)} = \frac{C_1(s+3) + C_2 s(s+3) + C_3 s^2}{s^2(s+3)} = \frac{(C_2 + C_3)s^2 + (C_1 + 3C_2)s + 3C_1}{s^2(s+3)}$$

Comparing the numerators we see that $C2 + C3 = 0$, $C1 + 3C2 = 0$, and $3C1 = 5$.

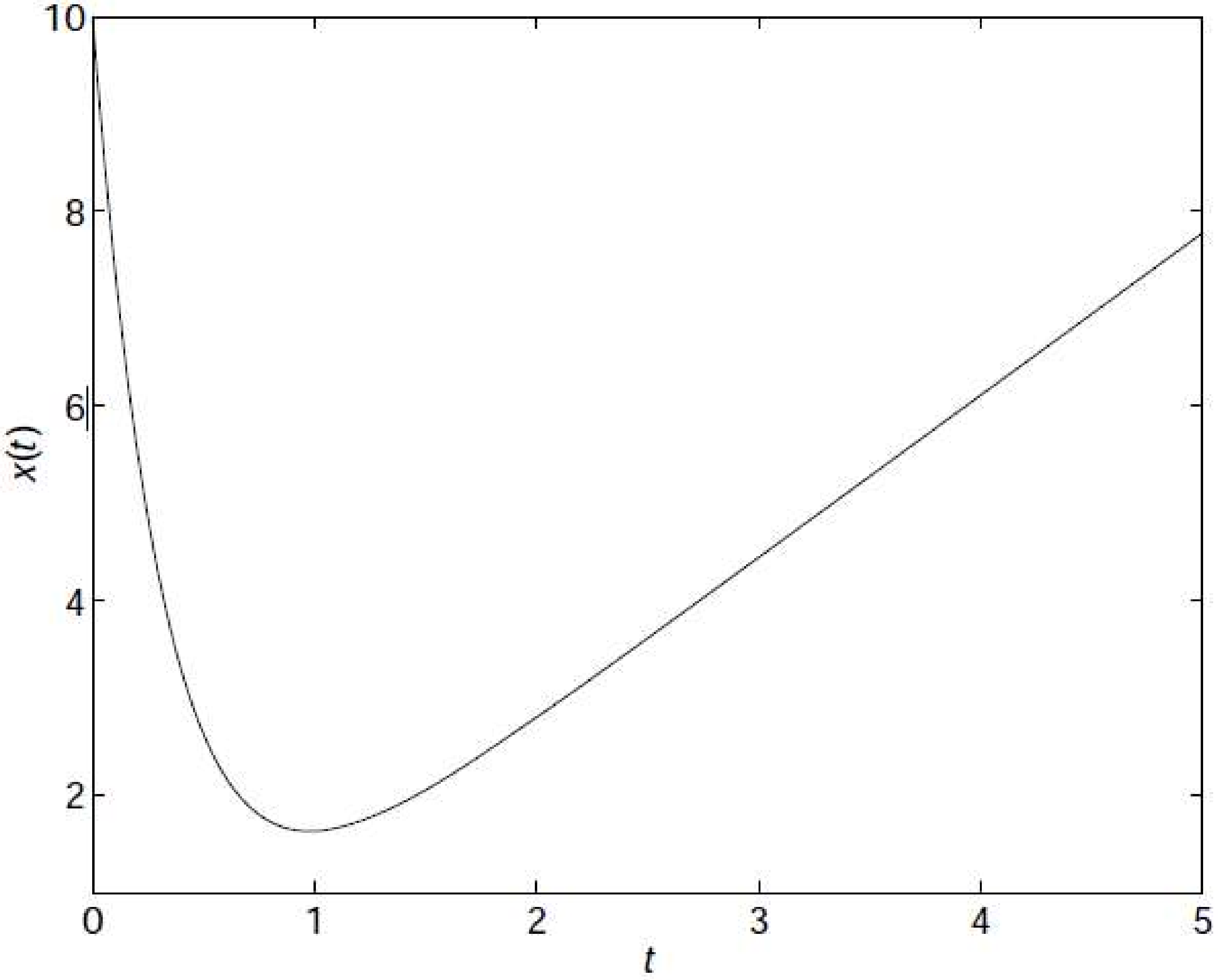
Thus, $C1 = 5/3$, $C2 = -C1/3 = -5/9$, and $C3 = -C2 = 5/9$

The forced response is $C_1 t + C_2 + C_3 e^{-3t} = \frac{5}{3}t - \frac{5}{9} + \frac{5}{9}e^{-3t}$

The complete response is the sum of the free and the forced response, and is

$$x(t) = 10e^{-3t} + \frac{5}{3}t - \frac{5}{9} + \frac{5}{9}e^{-3t}$$

The plot of the response is shown as follow.



10
8
6
4
2
x(t)
0
1
2
3
4
5
t

Homework

- Reading : Chapter 3.1-3.3