System modeling and simulation

Chapter 3. Solution Methods for Dynamic Models

3.3 Transfer Functions

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Definition for transfer function

- E.g. Consider the model

$$\dot{x} + ax = f(t) \quad x(0) = 0.$$

Transforming both sides of the equation gives

$$sX(s) + aX(s) = F(s)$$

Then solve for the ratio *X(s)/F(s)* and denote it by *T (s)*:

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{s+a}$$

T (s) : transfer function.

Transfer function?

- **The transfer function is the transform of forced response divided by the transform of the input.**

$$T(s) = \frac{X(s)}{F(s)}$$

Usage of transfer function

1. *Transfer Functions and Software.*

2. *ODE Equivalence.*

e.g. $$\frac{X(s)}{F(s)} = \frac{5}{s^2 + 7s + 10} \Rightarrow \ddot{x} + 7\dot{x} + 10x = 5f(t).$$

3. *The Transfer Function and Characteristic Roots.*

Multiple inputs and outputs

- Sometimes, models have more than one input or occur as sets of equations with more than one dependent variable.

- It is important to realize that there is one transfer function for each input-output pair.

Example 1

- Two Inputs and One Output:

Obtain the transfer functions *X(s)/F(s)* and *X(s)/G(s)* for the following equation.

$$5\ddot{x} + 30\dot{x} + 40x = 6\,f(t) - 20g(t)$$

Example 1

Using the derivative property with zero initial conditions, we can immediately write the equation as

$$5s^2X(s) + 30sX(s) + 40X(s) = 6F(s) - 20G(s)$$

Solve for *X(s)*.

$$X(s) = \frac{6}{5s^2 + 30s + 40}F(s) - \frac{20}{5s^2 + 30s + 40}G(s)$$

Example 1

When there is more than one input, the transfer function for a specific input can be obtained by temporarily setting the other inputs equal to zero. Thus, we obtain

$$\frac{X(s)}{F(s)} = \frac{6}{5s^2 + 30s + 40} \qquad \frac{X(s)}{G(s)} = -\frac{20}{5s^2 + 30s + 40}$$

Note that the denominators of both transfer functions have the same roots: s=−2 and s=−4.

Example 2

- A System of Equations

Obtain the transfer functions *X(s)/V(s)* and *Y (s)/V(s)* of the following system of equations:

$$\dot{x} = -3x + 2y$$

$$\dot{y} = -9y - 4x + 3v(t)$$

Example 2

- Solution: Here two outputs are specified, *x* and *y*, with one input, *v*. Thus there are two transfer functions.

To obtain them, transform both sides of each equation, assuming zero initial conditions.

$$sX(s) = -3X(s) + 2Y(s)$$

$$sY(s) = -9Y(s) - 4X(s) + 3V(s)$$

Example 2

These are two algebraic equations in the two unknowns, *X(s)* and *Y (s)*. Solve the first equation for *Y (s)*:

$$Y(s) = \frac{s+3}{2} X(s) \qquad (1)$$

Substitute this into the second equation.

$$s\frac{s+3}{2} X(s) = -9\frac{s+3}{2} X(s) - 4X(s) + 3V(s)$$

Example 2

Then solve for *X(s)/V(s)* to obtain

$$\frac{X(s)}{V(s)} = \frac{6}{s^2 + 12s + 35} \qquad (2)$$

Now substitute this into equation (1) to obtain

$$\frac{Y(s)}{V(s)} = \frac{s+3}{2}\frac{X(s)}{V(s)} = \frac{s+3}{2}\frac{6}{s^2 + 12s + 35} = \frac{3(s+3)}{s^2 + 12s + 35} \qquad (3)$$

Example 2

The desired transfer functions are given by equations (2) and (3).

Note that denominators of both transfer functions have the same factors, $s = -5$ and $s = -7$, which are the roots of the characteristic equation: $s2 + 12s + 35$.

Computing TRANSFER-FUNCTION With Matlab

tf and tfdata functions

e.g.

```
sys2 = tf([6, 9, 2],[5, 4, 7, 3])
```

$$5\frac{d^3x}{dt^3} + 4\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 3x = 6\frac{d^2f}{dt^2} + 9\frac{df}{dt} + 2f$$

Homework 2.1

- Reading through chapter 3.1-3.4.
- Problem 3.1, 3.4, 3.5, 3.9, 3.10, 3.12, 3.16, 3.18

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3.4 Partial-Fraction Expansion

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Partial-Fraction Expansion

- To solve a differential equation by using the Laplace transform, we must be able to obtain a function *x(t)* from its transform *X(s)*. This process is called *inverting* the transform.
- If *X(s)* is of the form as follow, the method of partial-fraction expansion can be used.

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

m<=n, assume an is unity.

- There are two cases to be considered.

1. All the roots are distinct;

2. Two or more roots are identical (repeated).

Distinct Roots Case

If all the roots are distinct, we can express *X(s)* in factored form as follows:

$$X(s) = \frac{N(s)}{(s + r_1)(s + r_2) \cdots (s + r_n)}$$

where the roots are $s = -r1, -r2, \ldots, -rn$. This form can be expanded as

$$X(s) = \frac{C_1}{s + r_1} + \frac{C_2}{s + r_2} + \cdots + \frac{C_n}{s + r_n}$$

Where

$$C_i = \lim_{s \to -r_i} [X(s)(s + r_i)]$$

Thus

$$x(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t} + \cdots + C_n e^{-r_n t}$$

- One Zero Root and One Negative Root:

e.g.

Obtain the inverse Laplace transform of

$$X(s) = \frac{5}{s(s+3)}$$

- The denominator roots are *s* =0 and *s*=−3, which are distinct and real. Thus the partial-fraction expansion has the form

$$X(s) = \frac{5}{s(s+3)} = \frac{C_1}{s} + \frac{C_2}{s+3}$$

Using the coefficient formula, we obtain

$$C_1 = \lim_{s\to 0}\left[s\frac{5}{s(s+3)}\right] = \lim_{s\to 0}\left[\frac{5}{(s+3)}\right] = \frac{5}{3}$$

$$C_2 = \lim_{s\to -3}\left[(s+3)\frac{5}{s(s+3)}\right] = \lim_{s\to -3}\left(\frac{5}{s}\right) = -\frac{5}{3}$$

The inverse transform is

$$x(t) = C_1 + C_2 e^{-3t} = \frac{5}{3} - \frac{5}{3}e^{-3t}$$

Repeated-Roots Case

- Suppose that *p* of the roots have the same value $s=-r1$, and the remaining *(n − p)* roots are distinct and real. Then *X(s)* is of the form

$$X(s) = \frac{N(s)}{(s+r_1)^p(s+r_{p+1})(s+r_{p+2})\cdots(s+r_n)}$$

The expansion is

$$X(s) = \frac{C_1}{(s+r_1)^p} + \frac{C_2}{(s+r_1)^{p-1}} + \cdots + \frac{C_p}{s+r_1} + \cdots + \frac{C_{p+1}}{s+r_{p+1}} + \cdots + \frac{C_n}{s+r_n}$$

Distinct Roots Case

If all the roots are distinct, we can express *X(s)* in factored form as follows:

$$X(s) = \frac{N(s)}{(s + r_1)(s + r_2) \cdots (s + r_n)}$$

where the roots are $s = -r1, -r2, \ldots, -rn$. This form can be expanded as

$$X(s) = \frac{C_1}{s + r_1} + \frac{C_2}{s + r_2} + \cdots + \frac{C_n}{s + r_n}$$

Where

$$C_i = \lim_{s \to -r_i} [X(s)(s + r_i)]$$

Thus

$$x(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t} + \cdots + C_n e^{-r_n t}$$

- The coefficients for the repeated roots are found from

$$C_1 = \lim_{s \to -r_1} \left[X(s)(s + r_1)^p\right]$$

$$C_2 = \lim_{s \to -r_1} \left\{\frac{d}{ds}\left[X(s)(s + r_1)^p\right]\right\}$$

$$\vdots$$

$$C_i = \lim_{s \to -r_1} \left\{\frac{1}{(i-1)!}\frac{d^{i-1}}{ds^{i-1}}\left[X(s)(s + r_1)^p\right]\right\} \qquad i = 1, 2, \ldots, p$$

The coefficients for the distinct roots are same as above.

The solution for the time function is

$$X(t) = C_1 \frac{t^{p-1}}{(p-1)!} e^{-r_1 t} + C_2 \frac{t^{p-2}}{(p-2)!} e^{-r_1 t} + \cdots + C_p e^{-r_1 t} + \cdots + C_{p+1} e^{-r_{p+1} t} + \cdots + C_n e^{-r_n t}$$

- One Negative Root and Two Zero Roots

e.g.

Obtain the inverse Laplace transform of

$$X(s) = \frac{5}{s^2(3s + 12)}$$

Solution:

The denominator roots are $s = -12/3 = -4$, $s = 0$, and $s = 0$. Thus the partial-fraction expansion has the form

$$X(s) = \frac{5}{s^2(3s + 12)} = \frac{1}{3}\frac{5}{s^2(s + 4)} = \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{C_3}{s + 4}$$

Using the coefficient formulas with *p* = 2 and *r*1 = 0, we obtain

$$C_1 = \lim_{s\to 0}\left[s^2 \frac{5}{3s^2(s+4)}\right] = \lim_{s\to 0}\left[\frac{5}{3(s+4)}\right] = \frac{5}{12}$$

$$C_2 = \lim_{s\to 0}\frac{d}{ds}\left[s^2 \frac{5}{3s^2(s+4)}\right] = \lim_{s\to 0}\frac{d}{ds}\left[\frac{5}{3(s+4)}\right] = \lim_{s\to 0}\left[-\frac{5}{3}\frac{1}{(s+4)^2}\right] = -\frac{5}{48}$$

$$C_3 = \lim_{s\to -4}\left[(s+4)\frac{5}{3s^2(s+4)}\right] = \lim_{s\to -4}\left(\frac{5}{3s^2}\right) = \frac{5}{48}$$

The inverse transform is

$$x(t) = C_1 t + C_2 + C_3 e^{-4t} = \frac{5}{12} t - \frac{5}{48} + \frac{5}{48} e^{-4t}$$

Also, we can use LCD to obtain the inverse Laplace transform.

With the LCD method we have

$$\frac{1}{3}\left[\frac{5}{s^2(s+4)}\right] = \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{C_3}{s+4} = \frac{C_1(s+4) + C_2 s(s+4) + C_3 s^2}{s^2(s+4)}$$

$$= \frac{(C_2 + C_3)s^2 + (C_1 + 4C_2)s + 4C_1}{s^2(s+4)}$$

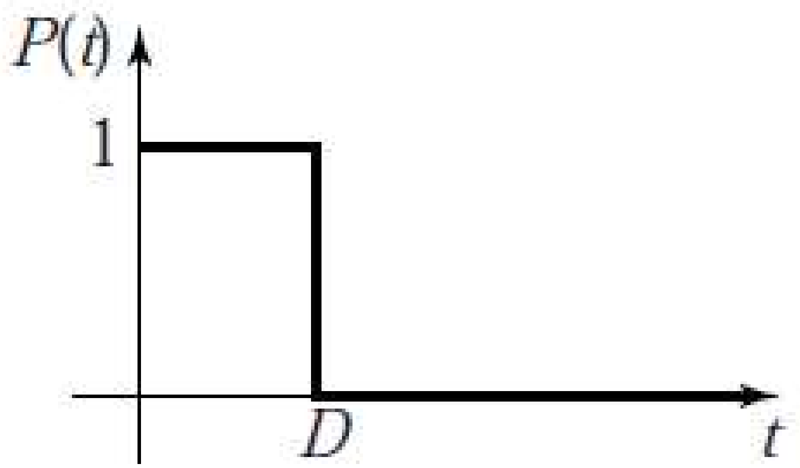
- Comparing numerators, we see that $C2 + C3 = 0$, $C1 + 4C2 = 0$, and $4C1 = 5/3$, which give $C1 = 5/12$, $C2 = -5/48$, and $C3 = 5/48$.

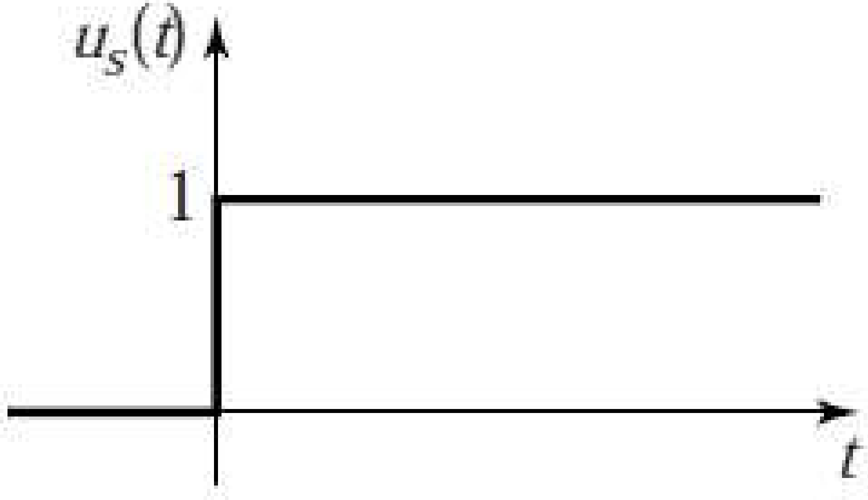
The inverse transform is

$$x(t) = C_1 t + C_2 + C_3 e^{-4t} = \frac{5}{12} t - \frac{5}{48} + \frac{5}{48} e^{-4t}$$

Impulse signal

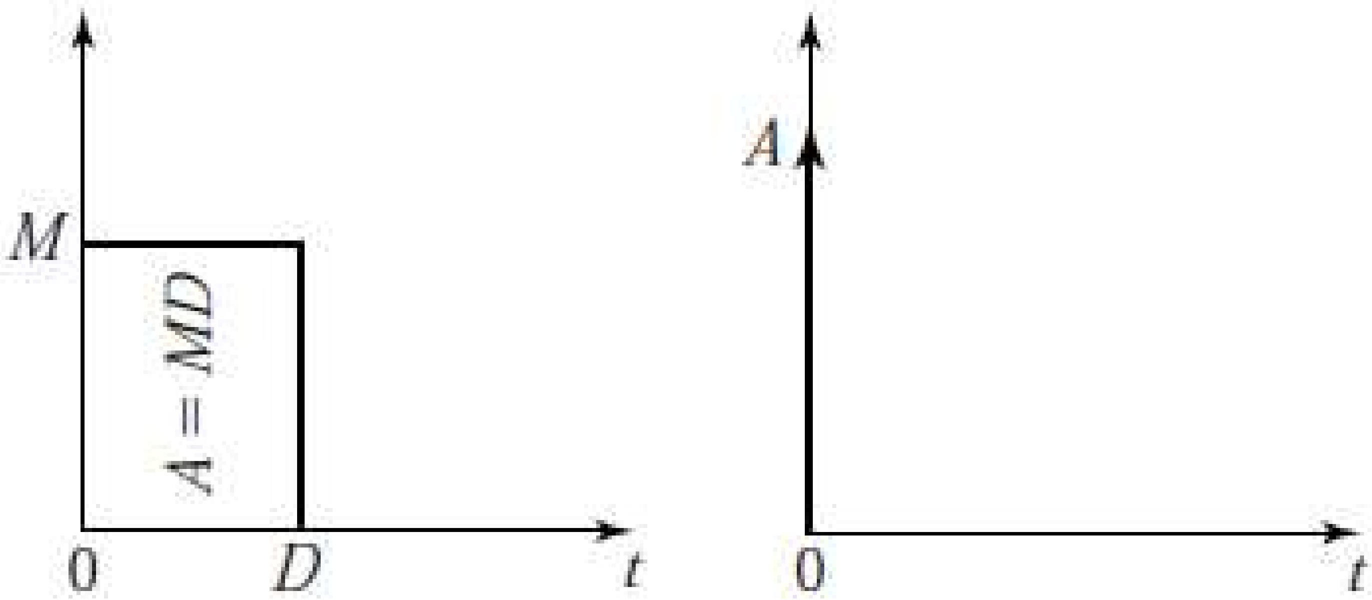
- Consider x(0) and x(0+)





Impulse signal

- Obtain transform of impulse signal



Impulse signal

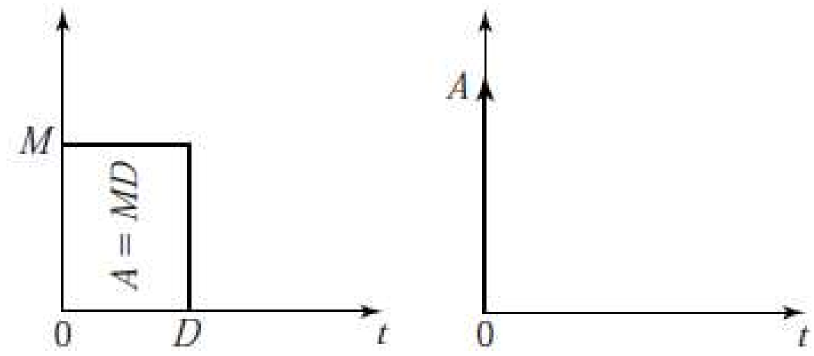
- Rectangle signal Laplace form: $M(1-e^{-sD})/s$
- D→0时， rectangle signal→impuls
- Laplace form:

$$F(s) = \lim_{D\to 0} \frac{A}{D}\frac{1-e^{-sD}}{s} = \lim_{D\to 0} \frac{Ase^{-sD}}{s} = A$$

- Dirac delta function:

$\delta(t)$

- Find a real process?



■ Problem

Obtain the unit-impulse response of the following model in two ways: (a) by separation of variables and (b) with the Laplace transform. The initial condition is $x(0) = 3$. What is the value of $x(0+)$?

$$\dot{x} = \delta(t)$$

■ Problem

Obtain the unit-impulse response of the following model. The initial conditions are $x(0) = 0$, $\dot{x}(0) = 0$. What are the values of $x(0+)$ and $\dot{x}(0+)$?

$$\frac{X(s)}{F(s)} = \frac{1}{2s^2 + 14s + 20}$$

Numerator dynamics

$$5\dot{x} + 10x = 2\dot{g}(t) + 10g(t)$$

- Consider if g(t) is a unit step signal?
- Discontinuity from 0 → 0+.

■ Problem

Obtain the transfer function and investigate the response of the following model in terms of the parameter a. The input $g(t)$ is a unit-step function.

$$3\ddot{x} + 18\dot{x} + 24x = a\dot{g}(t) + 6g(t) \qquad x(0) = 0 \qquad \dot{x}(0) = 0$$

Matlab script

```
[r,p,K] = residue(num,den)
```

e.g.

```
[r,p,K] = residue([4, 1],[1, 6, 34, 0])
```

$$X(s) = \frac{4s+1}{(s^2+6s+34)s} = \frac{4s+1}{s^3+6s^2+34s}$$

Homework 2.2

- Reading through chapter 3.5-3.10.
- Problem 3.19, 3.26, 3.50, 3.51