System Modeling and Simulation

Chapter 6. Electrical and Electromechanical Systems

6.1 Electrical Elements

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Electrical Elements

- Two primary variables used to describe a circuit's behavior

1. Current (A): $i = \frac{dQ}{dt}$ or $Q(t) = \int i\,dt$

2. Voltage(V): Energy is required to move a charge between two points in a circuit. The work per unit charge required to do this is called *voltage*.

$v = iR$ Ohm's law R is the resistance

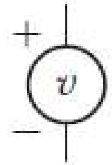
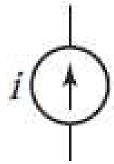
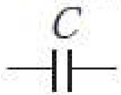
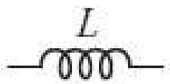
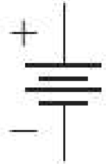
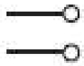
Active And Passive Elements

1. Passive elements: resistors; capacitors; inductors.

Elements that provide energy are *sources,* and elements that dissipate energy are *loads*.

2. Active elements: batteries; generators; thermocouples; solar cells.

Circuit symbols shown as follow.

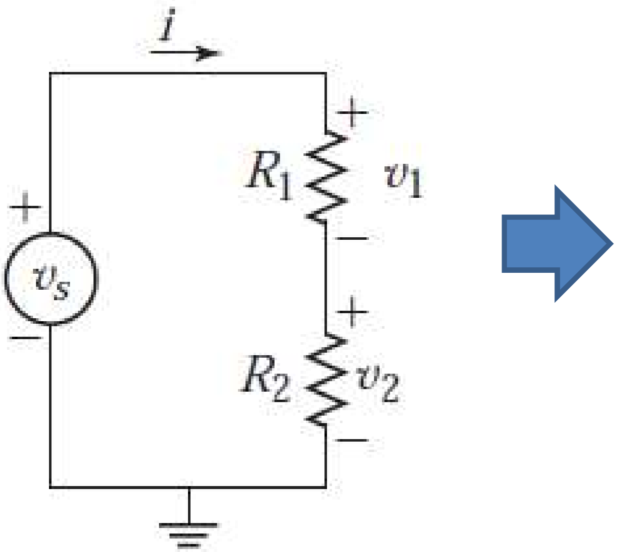
Quantity	Units	Circuit symbol
Voltage	volt (V)	Voltage Source 
Charge	coulomb (C) = N · m/V	
Current	ampere (A) = C/s	Current Source 
Resistance	ohm (Ω) = V/A	
Capacitance	farad (F) = C/V	
Inductance	henry (H) = V · s/A	
Battery	—	
Ground	—	
Terminals (input or output)	—	

Power P is work done per unit time, if the element is a linear resistor, the power is given by

$$P = iv = i^2 R = \frac{v^2}{R}$$

The Dynamics of Electrical Elements

1. Series Resistances



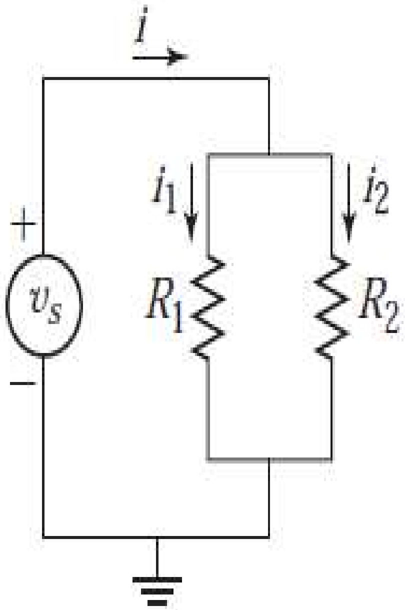
$$v_s = (R_1 + R_2)i$$

$$v_1 = R_1 i = \left(\frac{R_1}{R_1 + R_2}\right) v_s$$

$$v_2 = R_2 i = \left(\frac{R_2}{R_1 + R_2}\right) v_s$$

$$\frac{v_1}{v_2} = \frac{R_1}{R_2}$$ *voltage-divider rule.*

- 2. Parallel Resistances



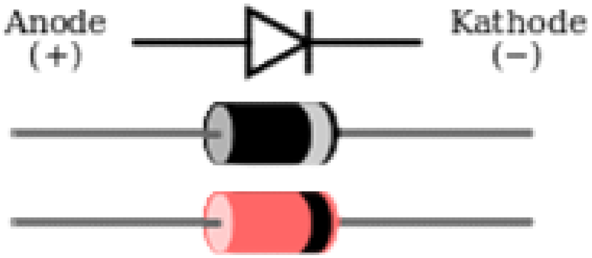
$$i = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) v_s$$

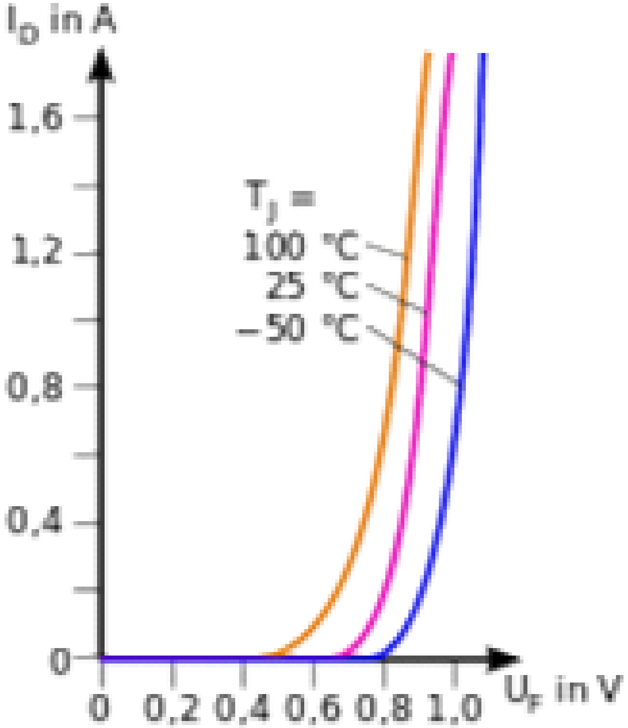
$$i_1 = \left(\frac{R_2}{R_1 + R_2}\right) i$$

$$i_2 = \left(\frac{R_1}{R_1 + R_2}\right) i$$

$$\frac{i_1}{i_2} = \frac{R_2}{R_1}$$

Nonlinear resistance





3. Capacitance: A *capacitor* is designed to store charge. The *capacitance C* of a capacitor is a measure of how much charge can be stored for a given voltage difference across the element.

$$v = Q/C = \frac{1}{C}\int i\,dt = \frac{1}{C}\int_0^t i\,dt + \frac{Q_0}{C}$$

Q_0 is the initial charge on the capacitor at time $t = 0$.

- In derivative form, this relation is expressed as

$$i = C\frac{dv}{dt}$$

4. Inductance:

The constitutive relation for an inductor is $\varphi = Li$, where L is the *inductance* and φ is the flux across the inductor.

The integral causality relation between flux and voltage is

$$\phi = \int v \, dt$$

Combining the two preceding expressions for φ gives the voltage-current relation for the inductor.

$$i = \frac{1}{L}\int v\,dt$$

which is equivalent to

$$v = L\frac{di}{dt}$$

The unit of inductance is the *henry* (H).

5. power and Energy:

Capacitors and inductors store electrical energy as stored charge and in a magnetic field, respectively.

The energy *E* stored in a capacitor can be found as follows:

$$E = \int P\,dt = \int iv\,dt = \int \left(C\frac{dv}{dt}\right) v\,dt = C\int v\,dv = \frac{1}{2}Cv^2$$

Similarly, the energy *E* stored in an inductor is

$$E = \int P\,dt = \int iv\,dt = \int i\left(L\frac{di}{dt}\right)dt = L\int i\,di = \frac{1}{2}Li^2$$

The follow table summarizes the voltage-current relations and the energy expressions for resistance, capacitance, and inductance elements.

Resistance:	$v = iR$	$P = Ri^2 = \frac{v^2}{R}$
Capacitance:	$v = \frac{1}{C}\int_0^t i\,dt + \frac{Q_0}{C}$	$E = \frac{1}{2}Cv^2$
Inductance:	$v = L\frac{di}{dt}$	$E = \frac{1}{2}Li^2$

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6.1 Circuit Examples

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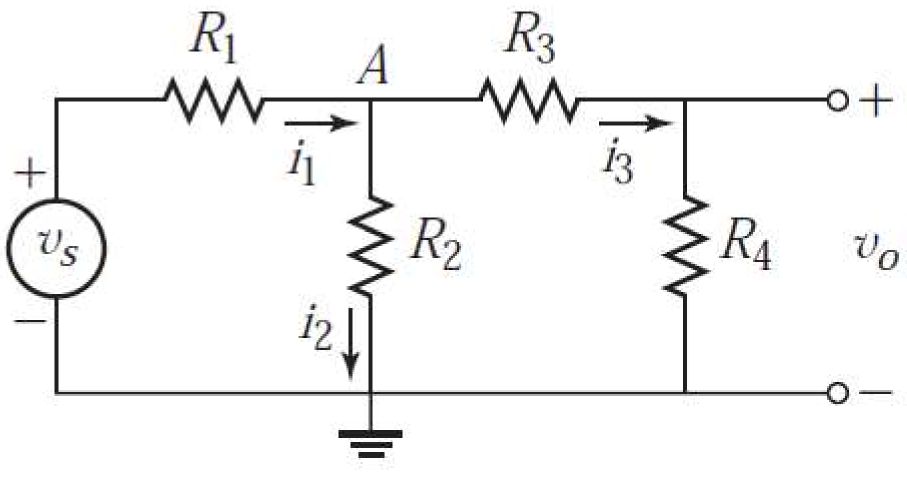
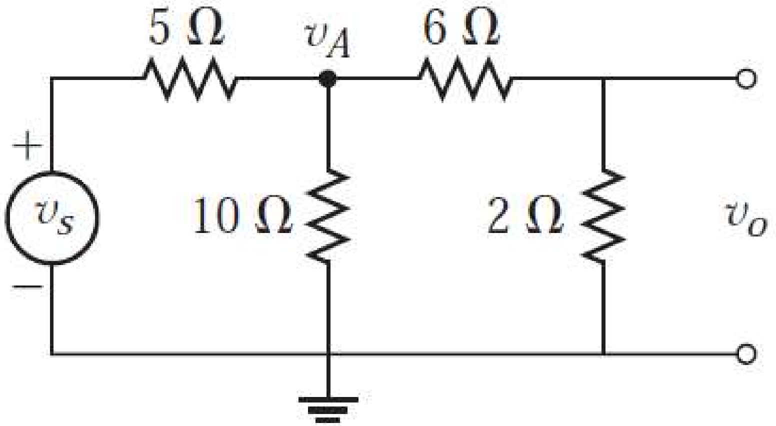
Circuit Examples

- 1. Application of the Voltage-Divider Rule

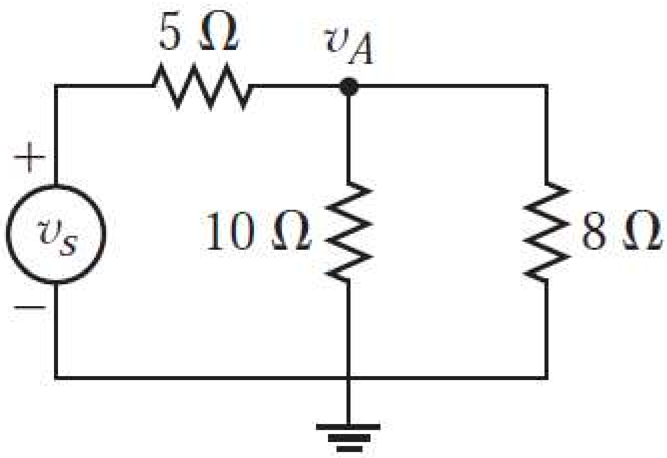
Consider the circuit shown in the Figure. Obtain the voltage *vo* as a function of the applied voltage *vs* by applying the voltage-divider rule. Use the values *R*1 = 5, *R*2 = 10, *R*3 = 6,and *R*4 = 2 .



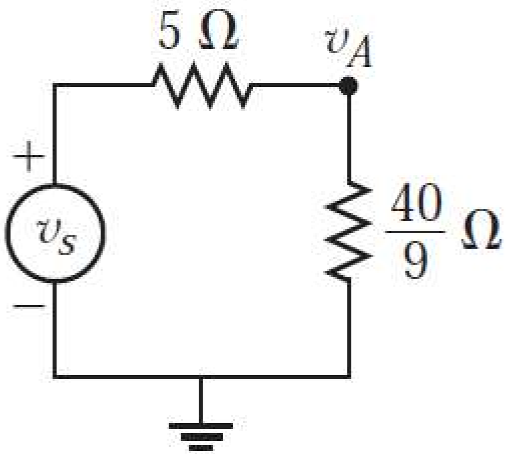
Application of the Voltage-Divider Rule



(a)



(b)



(c)

Solution: Let *vA* be the voltage at the node shown in Figure a.

$$v_o = \frac{R_4}{R_3 + R_4} v_A = \frac{2}{6+2} v_A = \frac{1}{4} v_A$$

Because resistors *R*3 and *R*4 are in series, we can add their values to obtain their equivalent resistance *Rs* = 2 + 6 = 8, as figure b.

Resistors *Rs* and *R*2 are parallel, so we can combine their values to obtain their equivalent resistance *Rp* as follows:

$$\frac{1}{R_p} = \frac{1}{10} + \frac{1}{8} = \frac{9}{40}$$

Thus, *Rp* = 40/9.

Finally, we apply the voltage-divider rule again to obtain

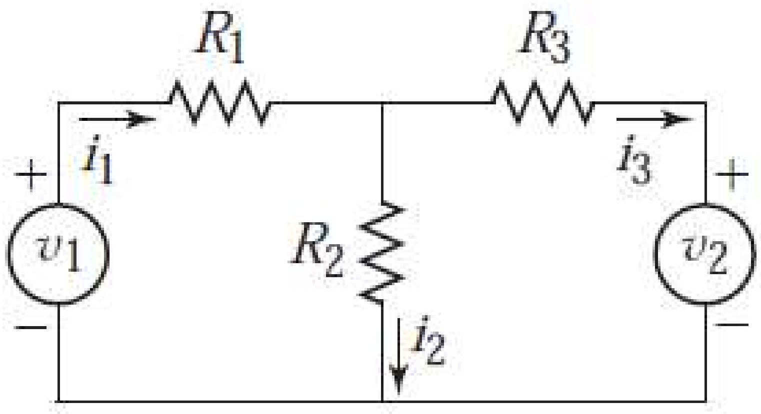
$$v_A = \frac{40/9}{5 + 40/9} v_s = \frac{8}{17} v_s$$

so

$$v_o = \frac{1}{4} v_A = \frac{1}{4}\left(\frac{8}{17}\right) v_s = \frac{2}{17} v_s$$

- 2. Analysis with Loop Currents

The values of the voltages and the resistances are given in the figure. Solve for the currents *i*1, *i*2, and *i*3 passing through the three resistors.



- Solution:

Applying Kirchhoff's voltage law to the left-hand loop gives

$$v_1 - R_1 i_1 - R_2 i_2 = 0$$

For the right-hand loop,

$$v_2 - R_2 i_2 + i_3 R_3 = 0$$

Combining $i_1 = i_2 + i_3.$

These are three equations in three unknowns.

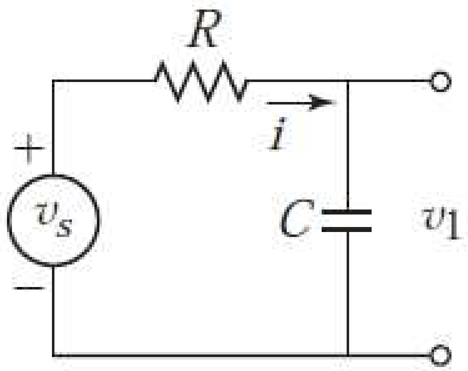
Their solution is

$$i_1 = \frac{(R_2 + R_3)v_1 - R_2 v_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$i_2 = \frac{R_3 v_1 + R_1 v_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

$$i_3 = \frac{R_2 v_1 - (R_1 + R_2)v_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

- 3. Series *RC* Circuit
- The resistor and capacitor in the circuit shown in the figure are said to be in series because the same current flows through them. Obtain the model of the capacitor voltage *v*1. Assume that the supply voltage *vs* is known.



From Kirchhoff's voltage law, $v_s - Ri - v_1 = 0$.
This gives $i = \frac{1}{R}(v_s - v_1)$ (1)

For the capacitor, $v_1 = \frac{1}{C}\int_0^t i\, dt + \frac{Q_0}{C}$

Differentiate this with respect to *t* to obtain

$$\frac{dv_1}{dt} = \frac{1}{C}i$$

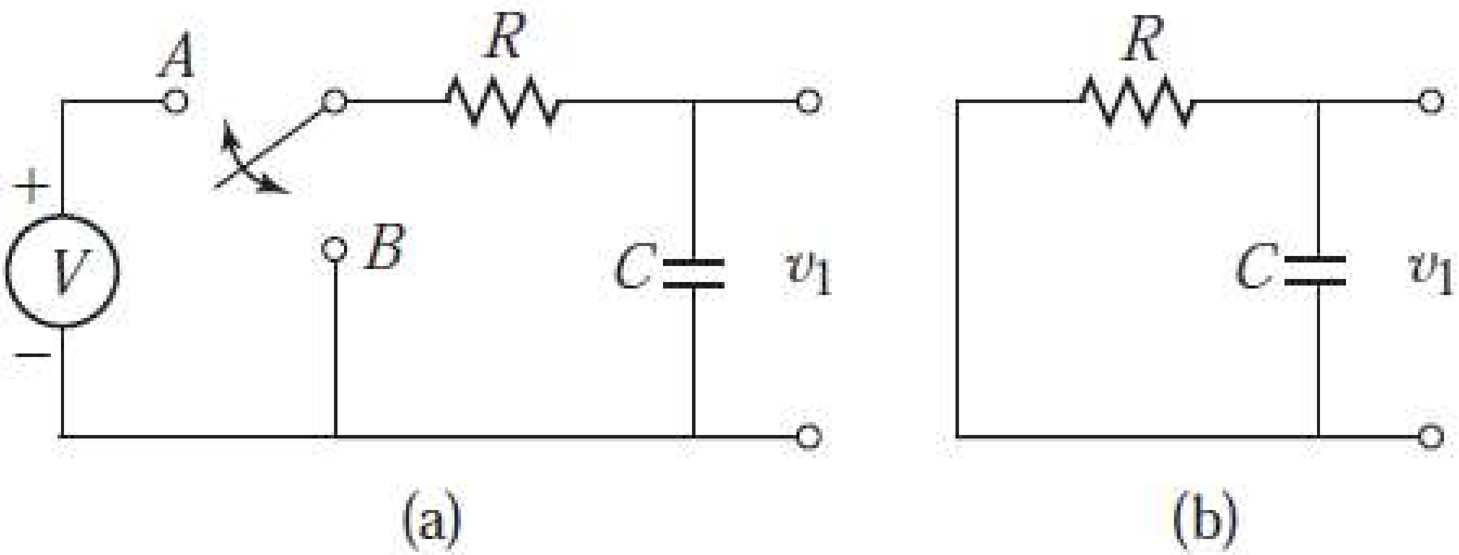
Then substitute for *i* from (1)

$$\frac{dv_1}{dt} = \frac{1}{RC}(v_s - v_1)$$

This the required model. It is often expressed in the following rearranged form:

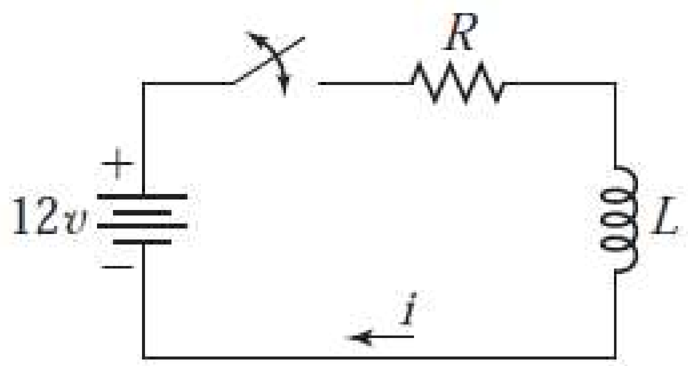
$$RC\frac{dv_1}{dt} + v_1 = v_s$$

Pulse response example



Telegraph

- R=20; L=4.



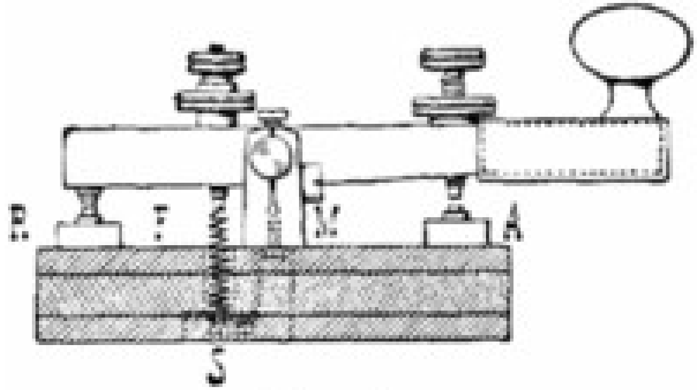
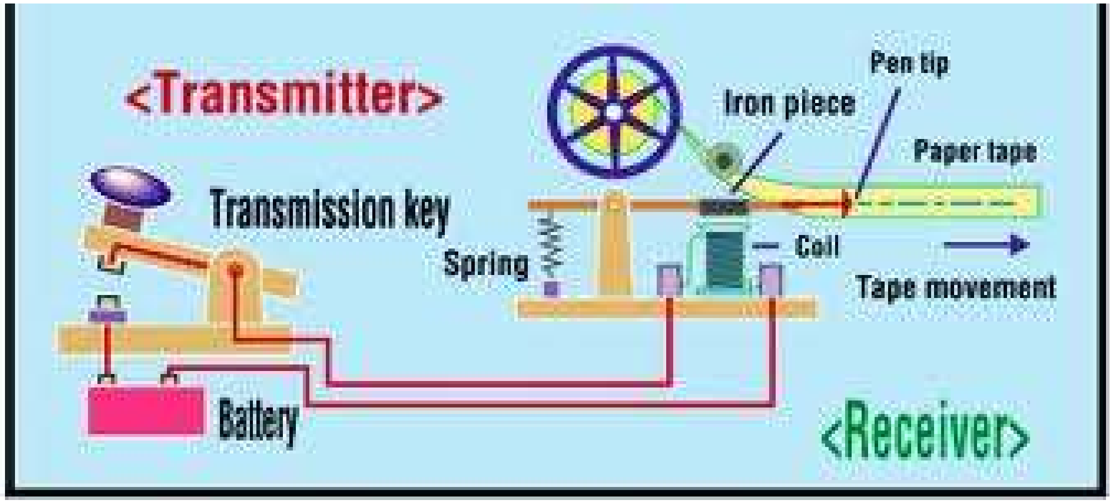
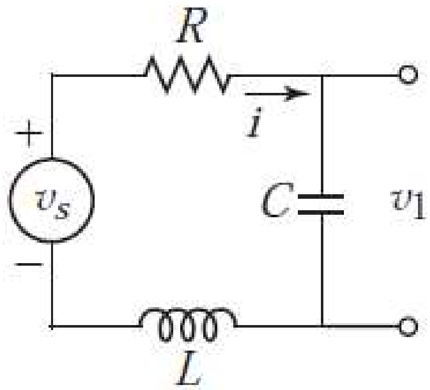


Fig. 6.



Series *RCL* Circuit

The resistor, inductor, and capacitor in the circuit shown in the figure are in series because the same current flows through them. Obtain the model of the capacitor voltage v_1 with the supply voltage v_s as the input.



Solution:

From Kirchhoff's voltage law,

$$v_s - Ri - L\frac{di}{dt} - v_1 = 0 \qquad (1)$$

For the capacıtor,

$$v_1 = \frac{1}{C}\int_0^t i\,dt$$

Differentiate this with respect to *t* to obtain

$$i = C\frac{dv_1}{dt}$$

substitute this for *i* in (1)

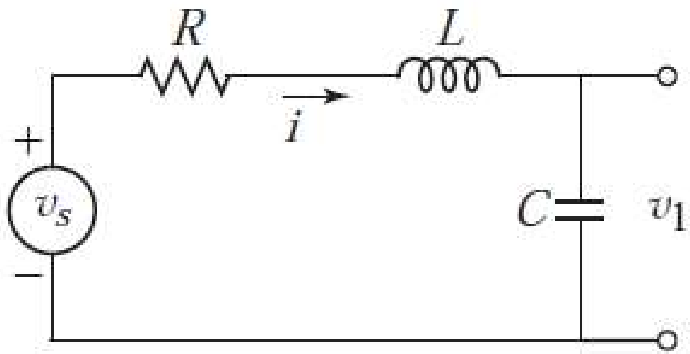
$$v_s - RC\frac{dv_1}{dt} - LC\frac{d^2v_1}{dt^2} - v_1 = 0$$

This is the required model. It can be expressed in the following form

$$LC\frac{d^2v_1}{dt^2} + RC\frac{dv_1}{dt} + v_1 = v_s$$

State-Variable Model of a Series *RLC* Circuit

Consider the series *RLC* circuit shown in the figure. Choose a suitable set of state variables, and obtain the state variable model of the circuit in matrix form. The input is the voltage v_s .



Solution:

In this circuit the energy is stored in the capacitor and in the inductor. The energy stored in the capacitor is $Cv_1^2/2$ and the energy stored in the inductor is $Li^2/2$. Thus a suitable choice of state variables is v_1 and i .

From Kirchhoff's voltage law,

$$v_s - Ri - L\frac{di}{dt} - v_1 = 0$$

Solve this for *di/dt*:

$$\frac{di}{dt} = \frac{1}{L}v_s - \frac{1}{L}v_1 - \frac{R}{L}i$$ the first state equation

Now find an equation for *dv*1*/dt*. From the capacitor relation,

$$v_1 = \frac{1}{C}\int i\,dt$$

Differentiating gives the second state equation.

$$\frac{dv_1}{dt} = \frac{1}{C}i$$

The two state equations can be expressed in matrix form as follows.

$$\begin{bmatrix} \dfrac{di}{dt} \\ \dfrac{dv_1}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i \\ v_1 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} v_s$$