# System modeling and simulation(ME340)

## Chapter 9. Transient response and block diagram model

## 9.1 Response of the first order system

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# Question?

- Different dynamics have been met till now, like mechanical, electrical, etc.
- Could we find the similarity of them?
- How do they response to the typical input?
- What are the typical inputs, impulse, step or ramp signals?

# Transfer function

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y$$

$$= b_m \frac{d^m f}{dt^m} + b_{m-1} \frac{d^{m-1} f}{dt^{m-1}} + \cdots + b_1 \frac{df}{dt} + b_0 f$$

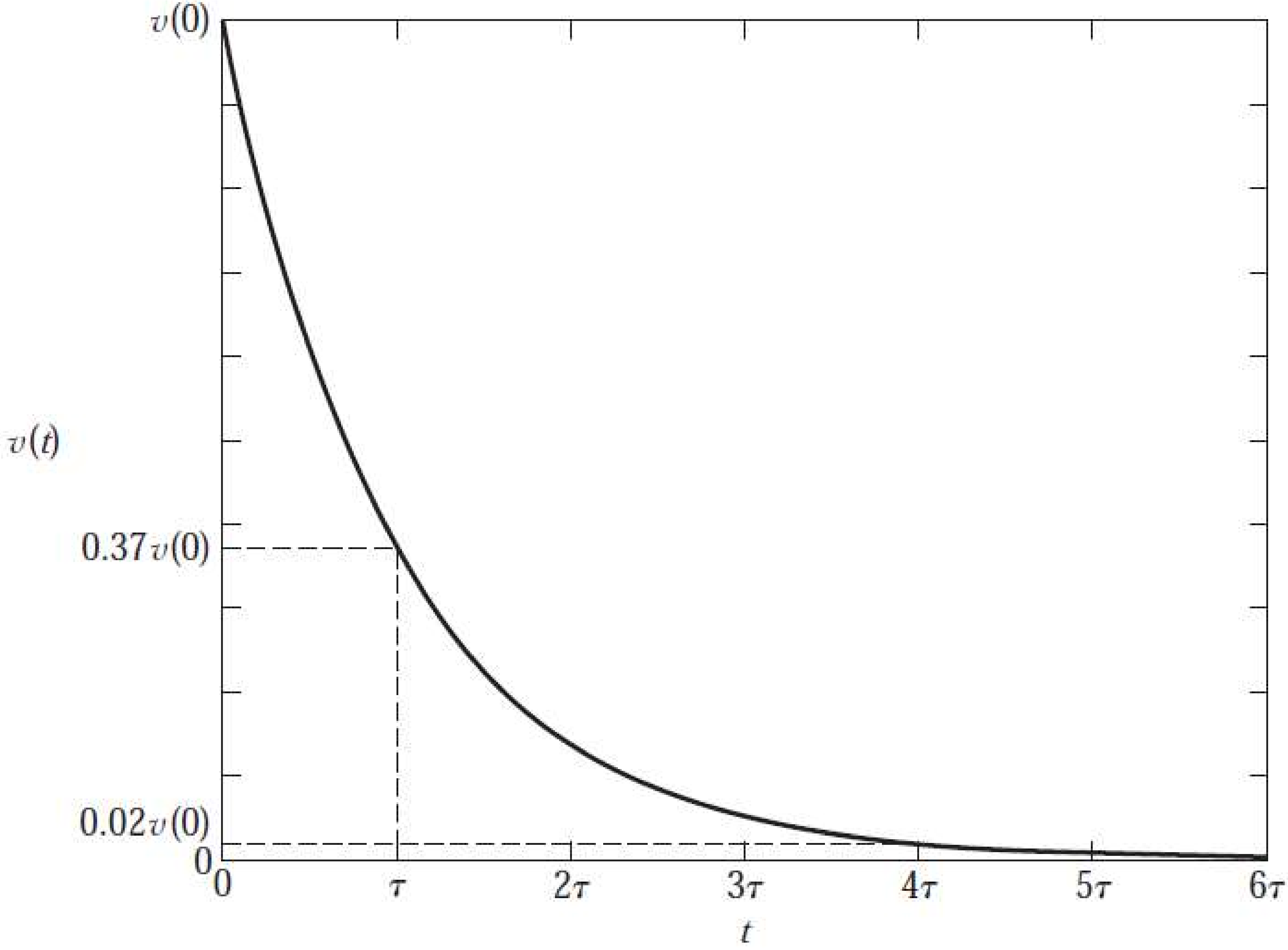
# Response of the first order systems

- $m\dot{v} + cv = f(t)$
- Free response :

$$v(t) = v(0)e^{-ct/m}$$

- The time constant: $\tau = \frac{m}{c}$

$$v(t) = v(0)e^{-t/\tau}$$



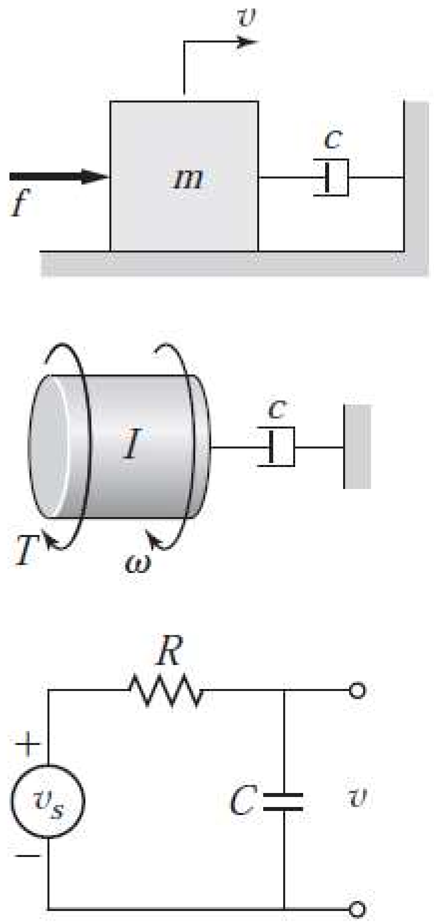
v(0)
v(t)
0.37v(0)
0.02v(0)
0
0
τ
2τ
3τ
4τ
5τ
6τ
t

# First order systems

(a)

(b)

(c)




$$m\frac{dv}{dt} + cv = f$$

$$\tau = \frac{m}{c}$$

$$I\frac{d\omega}{dt} + c\omega = T$$

$$\tau = \frac{I}{c}$$

$$RC\frac{dv}{dt} + v = v_s$$

$$\tau = RC$$

# Step response of a first-order model

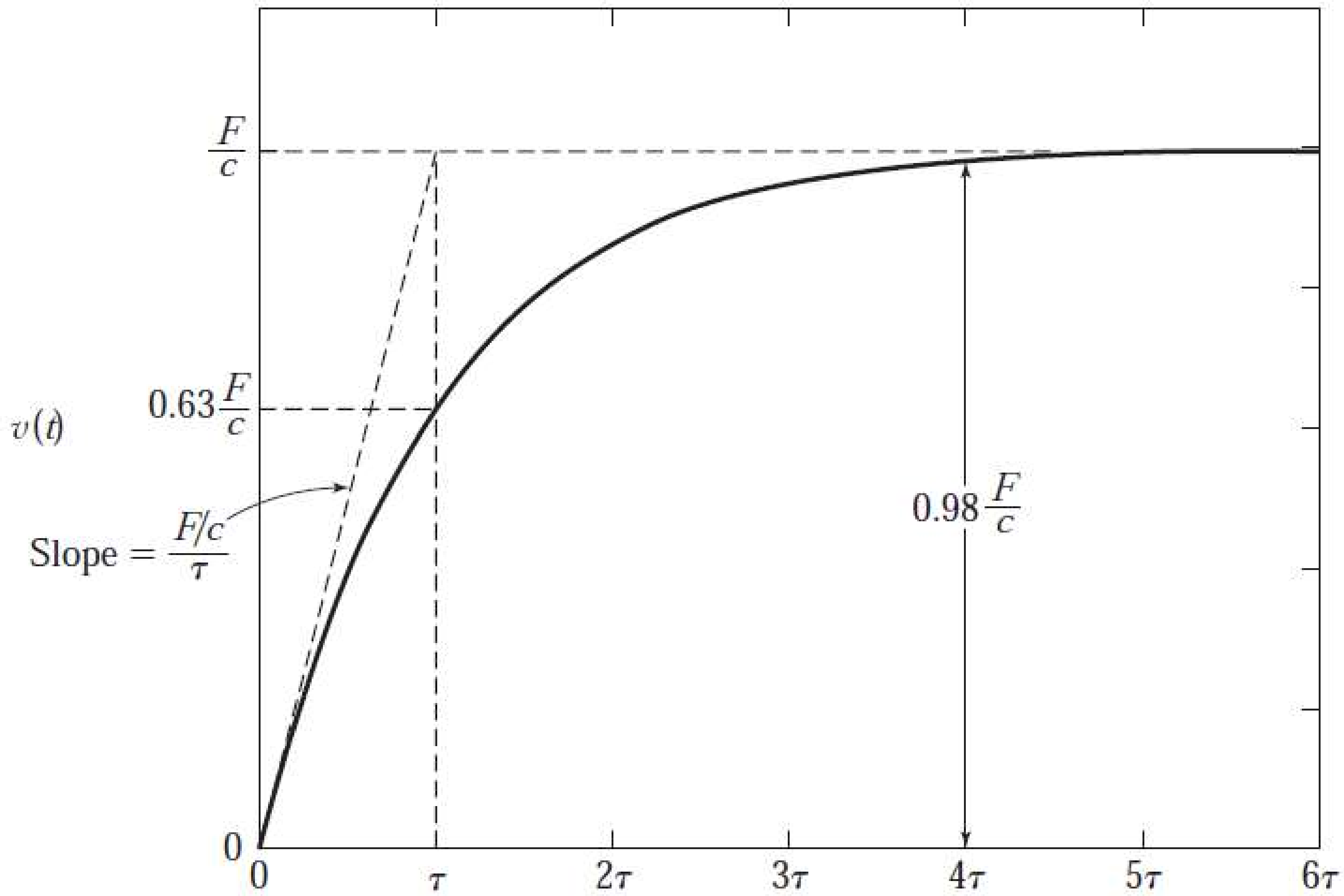


**Table 9.1.1** Free, step, and ramp response of $\tau\dot{y} + y = r(t)$.

Free response $[r(t) = 0]$

$y(t) = y(0)e^{-t/\tau}$

$y(\tau) \approx 0.37\, y(0)$

$y(4\tau) \approx 0.02\, y(0)$

Step response $[r(t) = Ru_s(t),\ y(0) = 0]$

$y(t) = R(1 - e^{-t/\tau})$

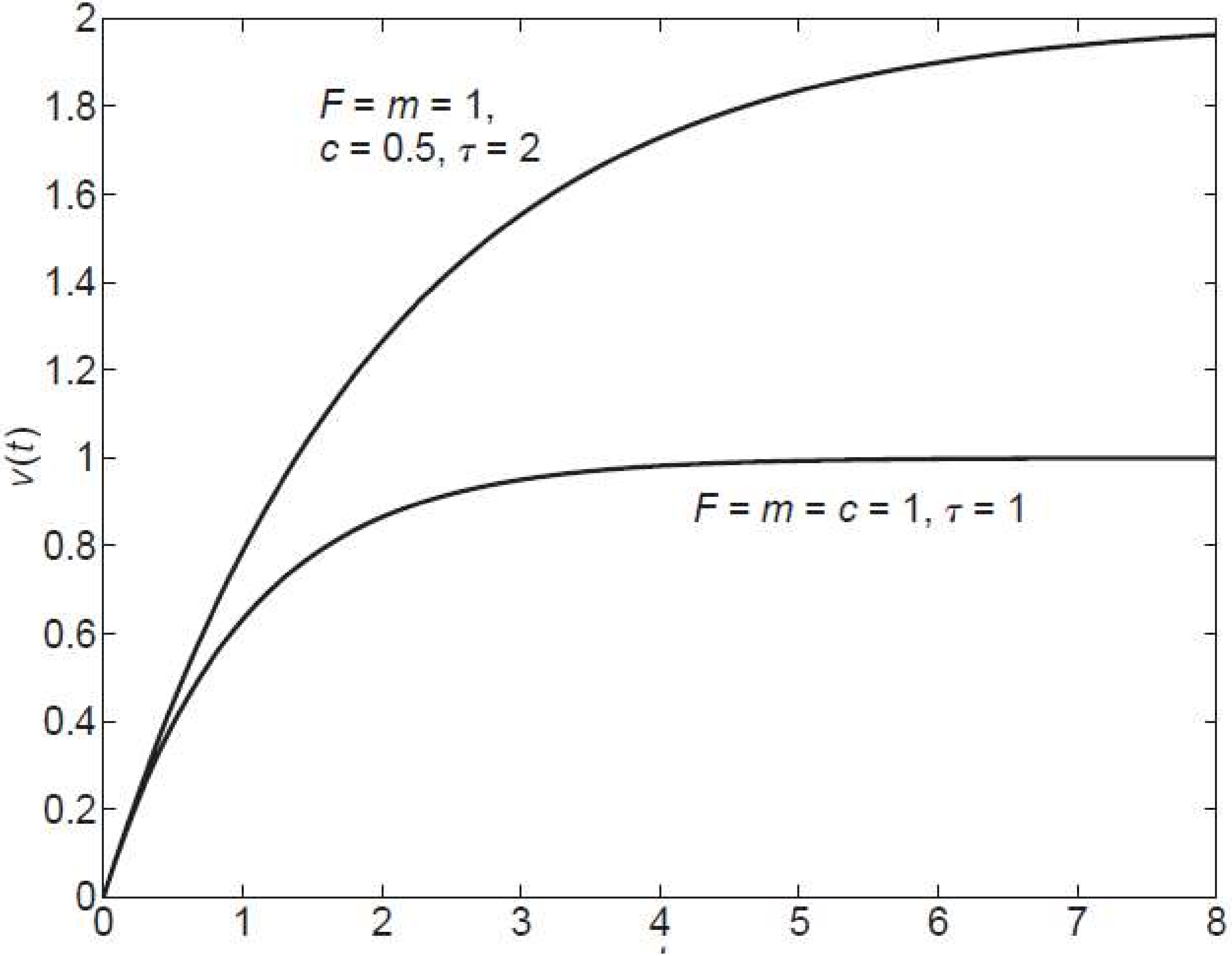
$y(\infty) = y_{ss} = R$

$y(\tau) \approx 0.63\, y_{ss}$

$y(4\tau) \approx 0.98\, y_{ss}$

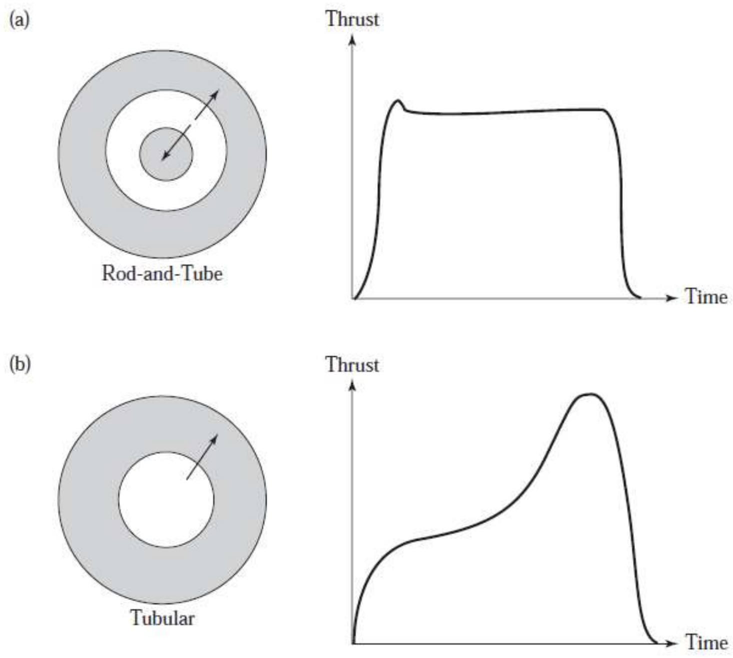
Ramp response $[r(t) = mt,\ y(0) = 0]$

$y(t) = m(t - \tau + \tau e^{-t/\tau})$

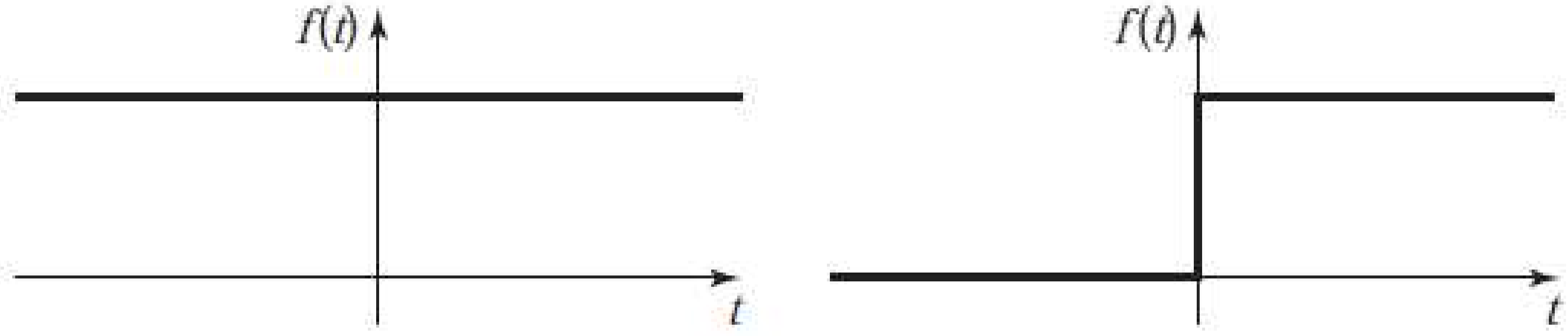

F = m = 1,
c = 0.5, τ = 2
F = m = c = 1, τ = 1
v(t)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
1
2
3
4
5
6
7
8

# The step function approximation

- The step function is an approximation of an input that can be switched on in the time interval that is very short compared to the time constant of the system.
- Step function



# Step inputs versus constant inputs




(a) (b)

# Step response with an input derivative

$$m\dot{v} + cv = b\dot{f}(t) + f(t)$$

# Impulse response

- A is impulse strength:

$$msV(s) - mv(0) + cV(s) = A$$

$$V(s) = \frac{mv(0) + A}{ms + c} = \frac{v(0) + A/m}{s + c/m}$$

$$v(t) = \left[v(0) + \frac{A}{m}\right] e^{-ct/m}$$

# Ramp response and time constant

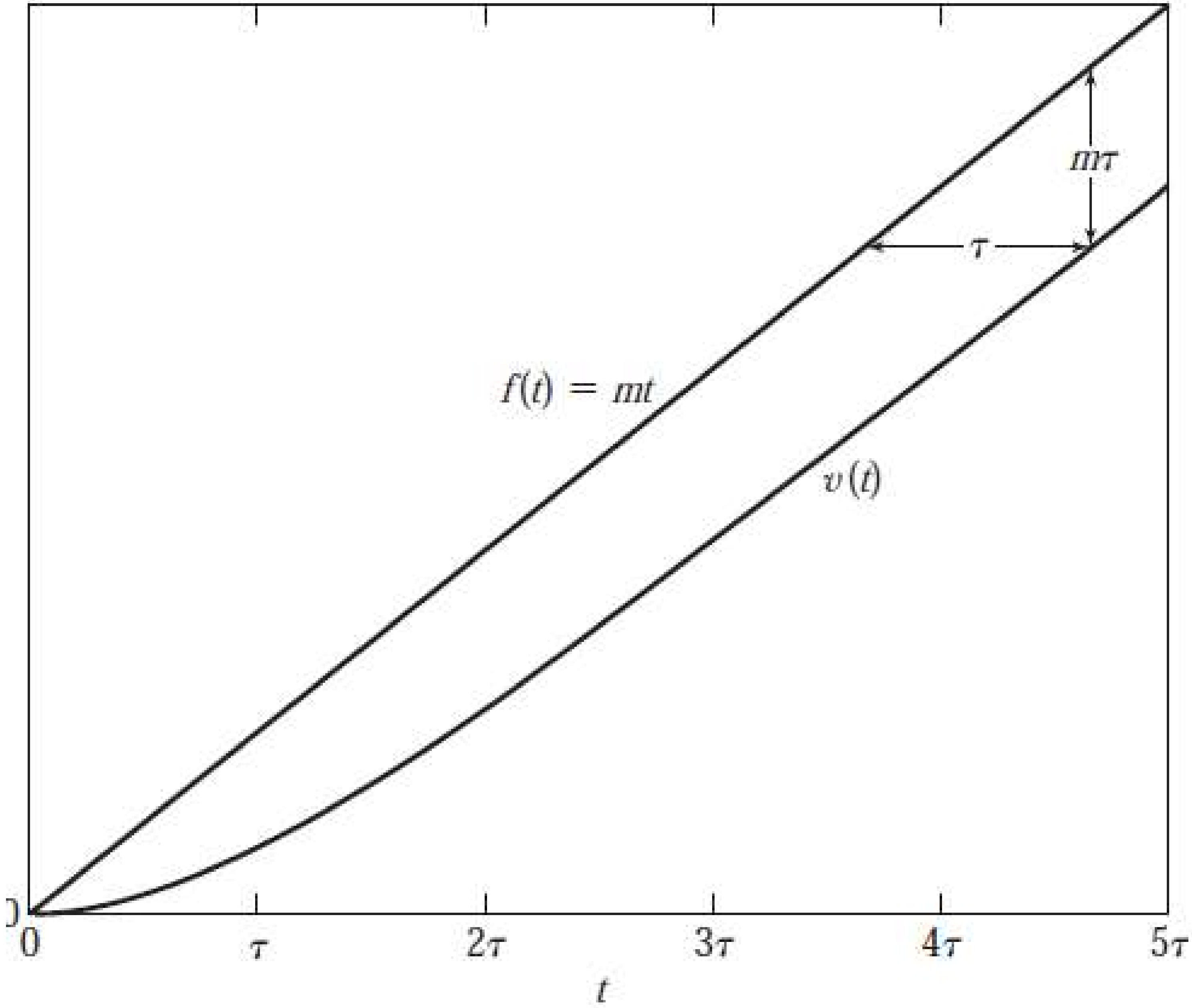
$$\tau\dot{v} + v = f(t)$$

$$f(t) = mt$$

$$\tau s V(s) + V(s) = F(s) = \frac{m}{s^2}$$

$$V(s) = \frac{m}{s^2(\tau s + 1)} = \frac{m}{s^2} - \frac{m\tau}{s} + \frac{m\tau}{s + 1/\tau}$$

$$v(t) = m(t - \tau) + m\tau e^{-t/\tau}$$



0
0
τ
2τ
3τ
4τ
5τ
t
f(t) = mt
v(t)
mτ
τ

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Chapter 9. Transient response and block diagram model

9.2 Response of the second order system

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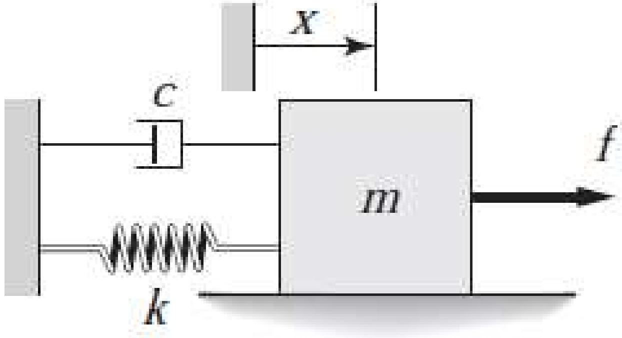
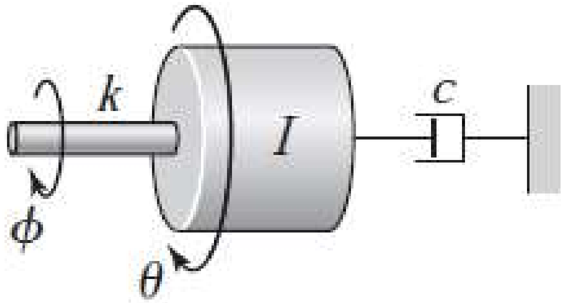
# Second order system transfer function

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

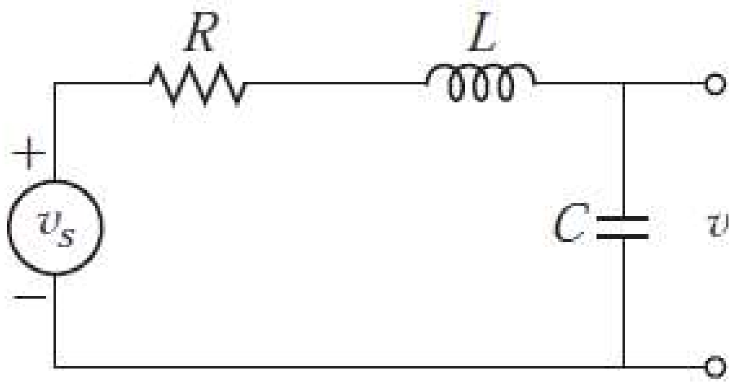
$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$




$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f$$




$$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\theta = k\phi$$




$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = v_s$$

# Response

- Free responses + forced responses
- Dependent of the characteristic roots

  $ms^2 + cs + k = 0$

  This model is stable when both of roots are real and negative or if the roots are complex with negative real parts.

  It is always true for reality.

# Second order system with numerator dynamics

$$m\ddot{x} + c\dot{x} + kx = a\dot{g}(t) + bg(t)$$

$$\frac{X(s)}{G(s)} = \frac{as + b}{ms^2 + cs + k}$$

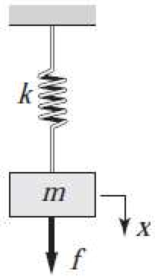
- The input does not influence the stability and characteristic equation

# Undamped response

- c=0:

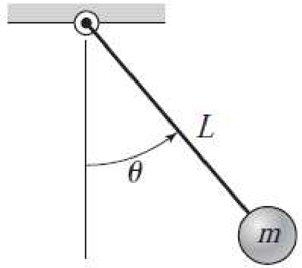
$$m\ddot{x} + kx = f(t)$$

- Natural frequency
- Period
- Amplitude of oscillation
- **Is this system stable?**



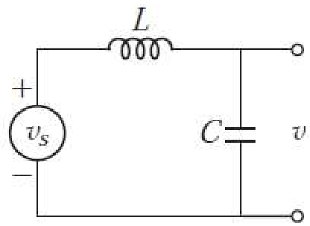

$$m\frac{d^2x}{dt^2} + kx = f(t)$$

$$\omega_n = \sqrt{\frac{k}{m}}$$




$$L\frac{d^2\theta}{dt^2} + g\theta = 0$$

$$\omega_n = \sqrt{\frac{g}{L}}$$

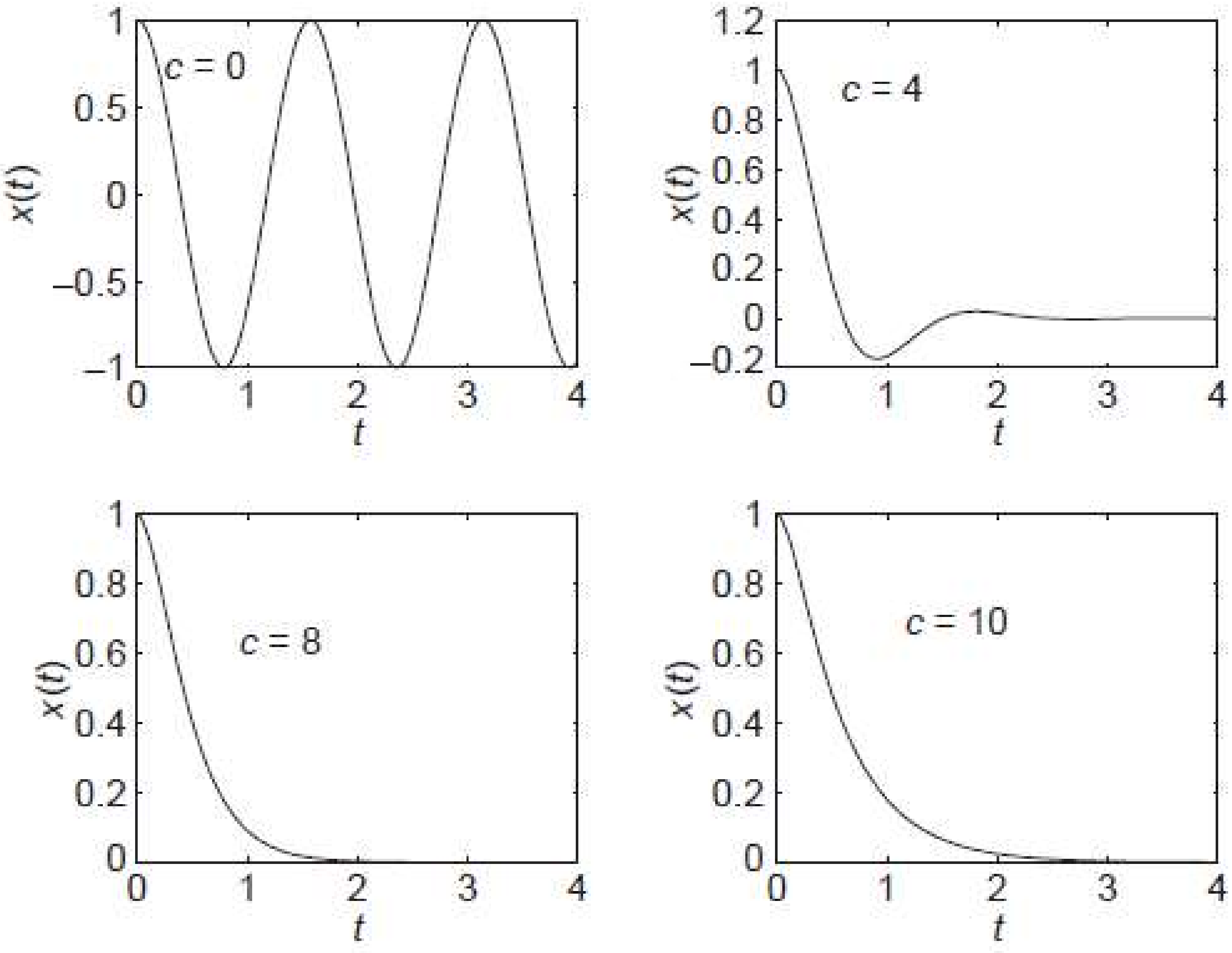



$$LC\frac{d^2v}{dt^2} + v = v_s$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$

# Consider different damping c

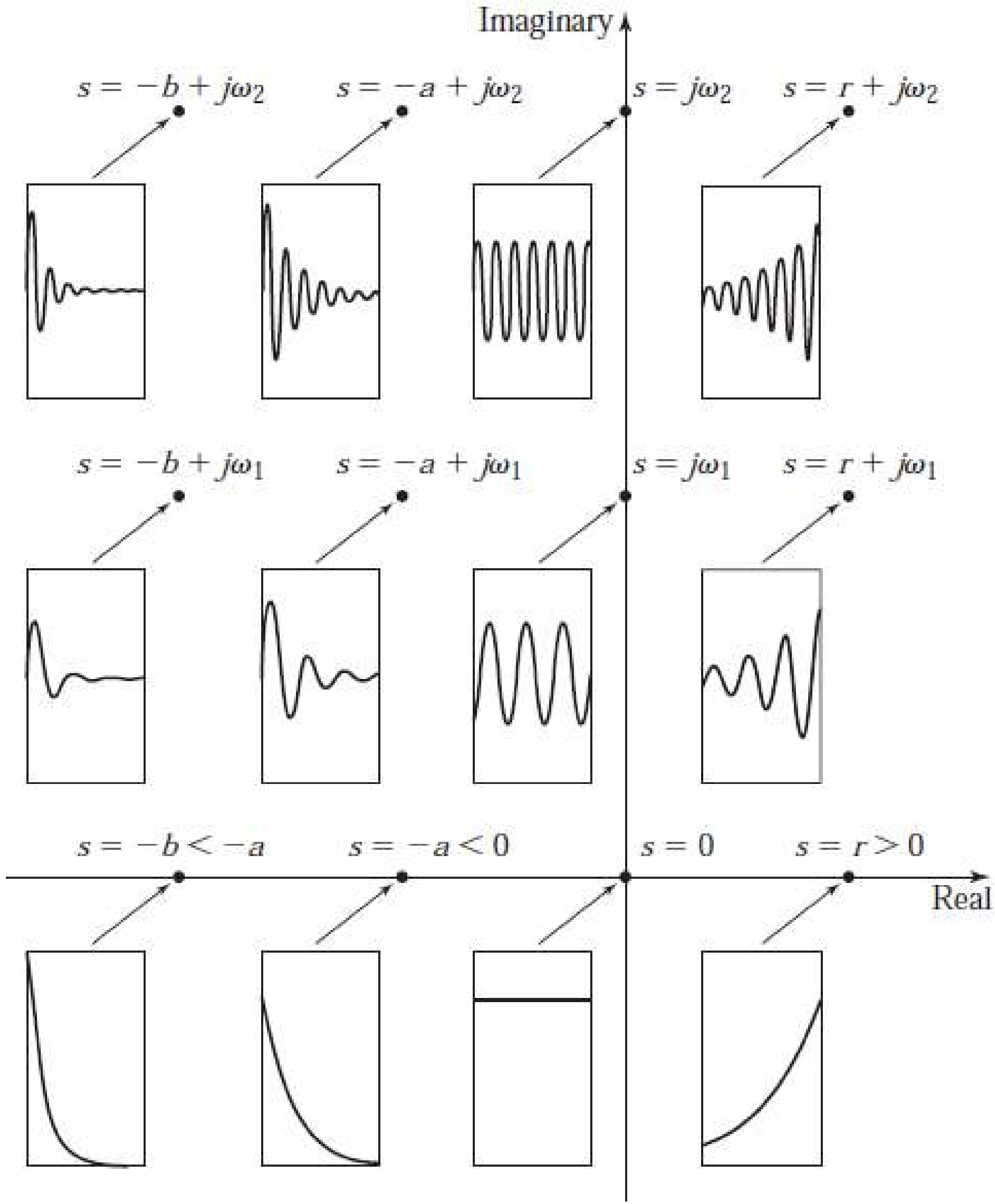
- m=1
- k=16



# Effect of root location

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = r \pm j\omega$$

1. Right of imaginary side
2. On the imaginary axis
3. Oscillate only non-zero imaginary part
4. Oscillation frequency
5. Decay



# The damping ratio

- Damping ratio in a second order system:

$$\zeta = \frac{c}{2\sqrt{mk}}$$

- This definition reveals the way the roots change from real to complex as the value c is changed.

# Examples

- For an unstable system damping ratio is meaningless.

$$3s^2 - 5s + 4 = 0$$

- Damping ratio can be used to check the oscillatory behavior.

$$s^2 + 5ds + 4d^2 = 0$$

# Natural and damped frequencies oscillation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

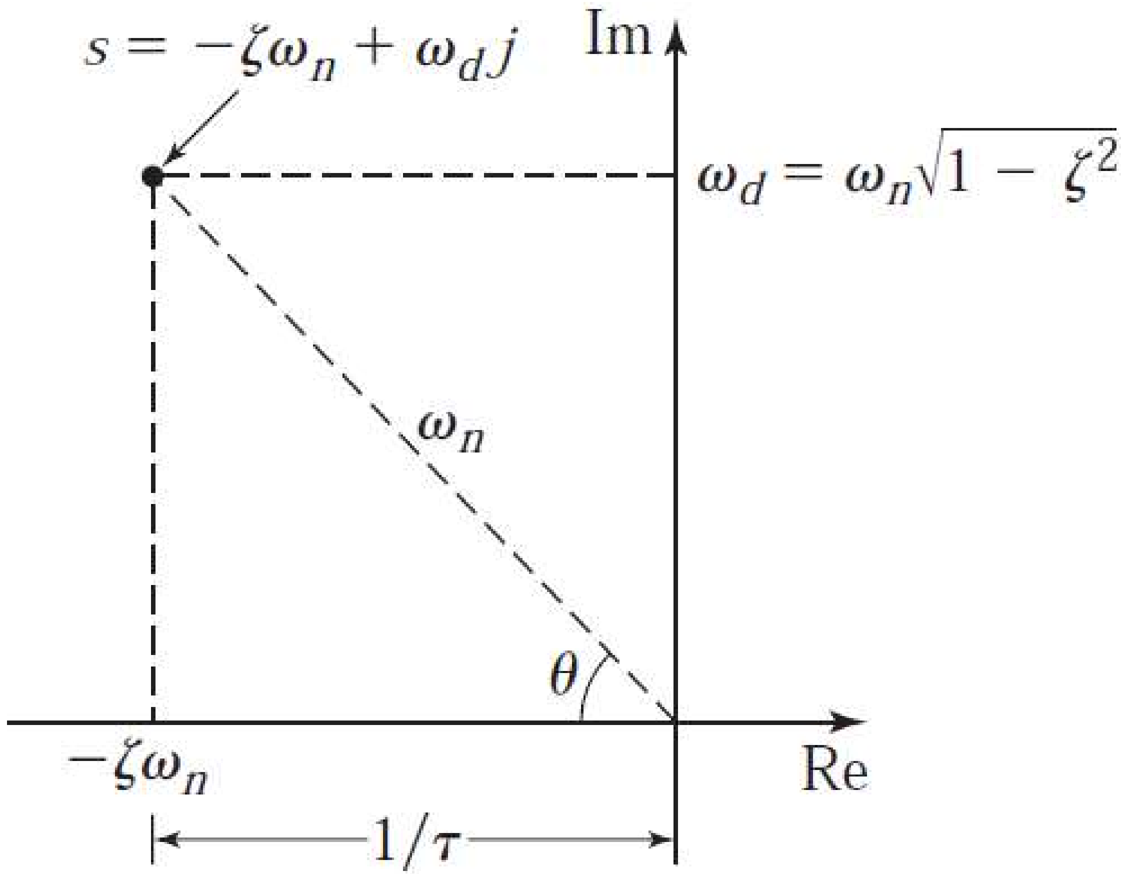
$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\tau = \frac{1}{\zeta\omega_n}$$

| | |
|---|---|
| Characteristic roots | $s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$ |
| Undamped natural frequency | $\omega_n = \sqrt{\dfrac{k}{m}}$ |
| Damping ratio | $\zeta = \dfrac{c}{2\sqrt{km}}$ |
| Damped natural frequency | $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ |
| Overdamped case ($\zeta > 1$) | Distinct, real roots: $s = -r_1, s = -r_2 \quad (r_1 \neq r_2)$ |
| | $x(t) = A_1 e^{-r_1 t} + A_2 e^{-r_2 t}$ |
| | $A_1 = \dfrac{\dot{x}(0) + r_2 x(0)}{r_2 - r_1}$ |
| | $A_2 = \dfrac{-r_1 x(0) - \dot{x}(0)}{r_2 - r_1} = x(0) - A_1$ |
| Critically damped case ($\zeta = 1$) | Repeated roots: $s = -r_1, s = -r_1$ |
| | $x(t) = (A_1 + A_2 t)\, e^{-r_1 t}$ |
| | $A_1 = x(0)$ |
| | $A_2 = \dot{x}(0) + r_1 x(0)$ |
| Underdamped case ($0 \le \zeta < 1$) | Complex conjugate roots: $s = -a \pm jb,\ b > 0$ |
| | $x(t) = De^{-at} \sin(bt + \phi)$ |
| | $D = +\dfrac{1}{b}\sqrt{[bx(0)]^2 + [\dot{x}(0) + ax(0)]^2}$ |
| | $\sin\phi = \dfrac{x(0)}{D} \quad \cos\phi = \dfrac{\dot{x}(0) + ax(0)}{bD}$ |
| Alternative form for $0 \le \zeta < 1$ | $x(t) = e^{-\zeta\omega_n t}\,[A \sin\omega_d t + x(0)\cos\omega_d t]$ |
| | $A = \dfrac{\zeta}{\sqrt{1-\zeta^2}} x(0) + \dfrac{1}{\omega_d}\dot{x}(0)$ |

# Graphical interpretation




$$\zeta = \cos\theta = \cos\left[\tan^{-1}(\tau\omega_d)\right]$$